On edge-disjoint empty triangles of point sets

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Abstract. Let P be a set of points in the plane in general position. Any three points $x, y, x \in P$ determine a triangle $\Delta(x, y, z)$ of the plane. We say that $\Delta(x, y, z)$ is empty if its interior contains no element of P. In this paper we study the following problems: What is the size of the largest family of edge-disjoint triangles of a point set? How many triangulations of P are needed to cover all the empty triangles of P? What is the largest number of edge-disjoint triangles of P containing a point q of the plane in their interior?

Introduction

Let P be a set of n points on the plane in general position. A *geometric graph* on P is a graph G whose vertices are the elements of P, two of which are adjacent if they are joined by a straight line segment. We say that G is *plane* if it has no edges that cross each other. A *triangle* of G consists of three points $x, y, z \in P$ such that xy, yz, and zx are edges of G; we will denote it as $\Delta(x, y, z)$. If in addition $\Delta(x, y, z)$ contains no elements of P in its interior, we say that it is *empty*.

In a similar way, we say that, if $x, y, z \in P$, then $\Delta(x, y, z)$ is a *triangle* of P, and that xy, yz, and zx are the *edges* of $\Delta(x, y, z)$. If $\Delta(x, y, z)$ is empty, it is called a 3-*hole* of P. A 3-hole of P can be thought of as an empty triangle of the complete geometric graph $\mathcal{K}_P$ on P. We remark that $\Delta(x, y, z)$ will denote a triangle of a geometric graph, and also a triangle of a point set.

A well-known result in graph theory says that, for $n = 6k + 1$ or $n = 6k + 3$, the edges of the complete graph K_n on n vertices can be decomposed into a set of $\binom{n}{2}/3$ edge-disjoint triangles. These decompositions are known as *Steiner triple systems* [**18**]; see also Kirkman's schoolgirl problem [**12, 17**]. In this paper, we address some variants of that problem, but for geometric graphs.

Given a point set P, let $\delta(P)$ be the size of the largest set of edge-disjoint empty triangles of P. It is clear that, if P is in convex position and it has $n = 6k + 1$ or $n = 6k + 3$ elements, then $\delta(P) = \binom{n}{2}/3$. On the other hand, we prove that, for some point sets, namely Horton point sets, $\delta(P)$ is $O(n \log n)$.

We then study the problem of covering the empty triangles of point sets with as few triangulations of P as possible. For point sets in convex position, we prove that we need essentially $\binom{n}{3}/4$ triangulations; our bound is tight. We also show that there are point

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sets P for which $O(n \log n)$ triangulations are sufficient to cover all the empty triangles of P for a given point set P.

Finally, we consider the problem of finding a point contained in the interior of many edge-disjoint triangles of P. We prove that for any point set there is a point contained in at least $n^2/12$ edge-disjoint triangles. Furthermore, any point in the plane is contained in at most $n^2/9$ edge-disjoint triangles of P, and this bound is sharp. In particular, we show that this bound is attained when P is the set of vertices of a regular polygon.

Preliminary work

The study of counting and finding k-holes in point sets has been an active area of research since Erdős and Szekeres [**6, 7**] asked about the existence of k-holes in planar point sets. It is known that any point set with at least ten points contains 5-holes; e.g. see [**9**]. Horton [**10**] proved that for $k \geq 7$ there are point sets containing no k-holes. The question of the existence of 6-holes remained open for many years, but recently Nicolás [**14**] proved that any point set with sufficiently many points contains a 6-hole. A second proof of this result was subsequently given by Gerken [**8**].

The study of properties of the set of triangles generated by point sets on the plane has been of interest for many years. Let $f_k(n)$ be the minimum number of k-holes that a point set has. Clearly a point set has a minimum of $f_3(n)$ empty triangles. Katchalski and Meir [**11**] proved that $\binom{n}{2} \leq f_3(n) \leq kn^2$ for some $k < 200$; see also Purdy [**16**]. Their lower bounds were improved by Dehnhardt [**4**] to $n^2 - 5n + 10 \leq f_3(n)$. He also proved that $\binom{n-3}{2} + 6 \leq f_4(n)$. Point sets with few k-holes for $3 \leq k \leq 6$ were obtained by Bárány and Valtr [**2**]. The interested reader can read [**13**] for a more accurate picture of the developments in this area of research.

Chromatic variants of the Erdős-Szekeres problem have recently been studied by Devillers, Hurtado, Károly, and Seara [**5**]. They proved among other results that any bi-chromatic point set contains at least $\frac{n}{4} - 2$ compatible monochromatic empty triangles. Aichholzer *et al.* [**1**] proved that every bi-chromatic point set contains $\Omega(n^{5/4})$ empty monochromatic triangles; this bound was improved by Pach and Tóth [**15**] to $\Omega(n^{4/3})$. Due to lack of space, we will omit the proofs of all of our results.

1 Sets of edge-disjoint empty triangles in point sets

Let P be a set of n points on the plane, and $\delta(P)$ the size of the largest set of edge-disjoint empty triangles of the complete graph $\mathcal{K}(P)$ on P. For any integer $k \geq 1$, let H_k denote the Horton set with 2^k points; see [**10**]. We will prove:

Theorem 1.1. *Let $n = 2^k$, and let H_k be the Horton set with $n = 2^k$ elements. Then $\delta(H_k)$ is $O(n \log n)$.*

Conjecture 1.2. *Every point set P in general position with n elements contains a set with at least $O(n \log n)$ edge-disjoint empty triangles.*

2 Covering the triangles of point sets with triangulations

An empty triangle t of a point set P is *covered* by a triangulation T of P if one of the faces of T is t. In this section we consider the following problem:

Problem 2.1. *How many triangulations of a point set are needed so that each empty triangle of P is covered by at least one triangulation?*

We start by studying Problem 2.1 for point sets in convex position, and then for point sets in general position. We will prove first:

Theorem 2.2. *The set of triangles of any convex polygon can be covered with*

(1) $\frac{1}{4}\left[\binom{n}{3} + \frac{n(n-2)}{2}\right]$ *triangulations for n even, and*

(2) $\frac{1}{4}\left[\binom{n}{3} + \frac{n(n-1)}{2}\right]$ *triangulations for n odd.*

This bound is tight.

Thus the number of triangulations needed to cover all the triangles of P is asymptotically $\binom{n}{3}/4$. The next result follows trivially:

Corollary 2.3. *Let P be a set of n points in convex position, and p any point in the interior of $CH(P)$. Then p belongs to the interior of at most $\frac{1}{4}\binom{n}{3} + O(n^2)$ triangles of P.*

Next we prove:

Theorem 2.4. $\Theta(n \log n)$ *triangulations of H_k are necessary and sufficient to cover the set of empty triangles of H_k.*

Conjecture 2.5. *At least $\Omega(n \log n)$ triangulations are needed to cover all the empty triangles of any point set with n points.*

3 A point in many edge-disjoint triangles

The problem of finding a point contained in many triangles of a point set was solved by Boros and Füredi [**3**]. They proved:

Theorem 3.1. *For any set P of n points in general position, there is a point in the interior of the convex hull of P contained in $\frac{2}{9}\binom{n}{3} + O(n^2)$ triangles of P. The bound is tight.*

We consider the following problem:

Problem 3.2. *Let P be a set of points on the plane in general position, and $q \notin P$ a point of the plane. What is the largest number of edge-disjoint triangles of P such that q belongs to the interior of all of them?*

We will prove:

Theorem 3.3. *In any point set in general position there is a point q for which the inequalities $\frac{1}{12}n^2 \leq \tau(q) \leq \frac{1}{9}n^2$ hold. Moreover, $\tau(q) \leq \frac{1}{9}n^2$ for every q.*

3.1 Regular polygons

By Theorem 3.3, any point in the interior of the convex hull of a point set is contained in at most $n^2/9$ edge-disjoint triangles of P. We now show that the upper bound in Theorem 3.3 is achieved when P is the set of vertices of a regular polygon. Proving this result proved to be a nice challenging problem. In what follows, we will assume that $n = 9m$ with $m \geq 1$. We will prove:

Theorem 3.4. *Let P be the set of vertices of a regular polygon with $n = 9m$ vertices, and let c be its center. If m is odd, then $|\tau(c)| \geq \frac{1}{9}n^2$, and if m is even, then $|\tau(c)| \geq \frac{1}{9}n^2 - n$.*

We conclude our paper by proving:

Theorem 3.5. *There are point sets P such that every $q \notin P$ is contained in at most a linear number of empty edge-disjoint triangles of P. This bound is tight.*

We conclude with the following:

Conjecture 3.6. *Let P be a set of n points in general position on the plane. Then there is a point q on the plane which is contained in at least $\log n$ edge-disjoint triangles of P.*

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