Rectilinear convex hull with minimum area

Carlos Alegría-Galicia[3], **Tzolkin Garduño**[3], **Areli Rosas-Navarrete**[3], **Carlos Seara**[2], **Jorge Urrutia**[1]

[1] Instituto de Matemáticas, Universidad Nacional Autónoma de México (UNAM)
urrutia@matem.unam.mx

[2] Universitat Politècnica de Catalunya (UPC)
carlos.seara@upc.edu

[3] Posgrado Ciencia e Ingeniería de la Computación, Universidad Nacional Autónoma de México (UNAM)
{alegria_c,garduno_t,areli}@uxmcc2.iimas.unam.mx

Abstract. Let P be a set of n points in the plane. We solve the problem of computing the orientations for which the rectilinear convex hull of P has minimum area in optimal $\Theta(n \log n)$ time and $O(n)$ space.

Introduction

The interest in the rectilinear convex hull of planar point sets arises from the study of ortho-convexity [**10**], a relaxation of traditional convexity. Unlike convex regions, an ortho-convex region might be disconnected, which makes the study of the ortho-convex closure for a point set [**5, 8**] harder. Several definitions have been presented by different authors. We will use a definition stated by Ottman et al. [**8**] as the *mr-convex hull*, see also Matoušěk et al. [**5, 7**]. The study of rectilinear convex hulls has gained attention partly because of some applications in digital image processing [**3**] and VLSI circuit layout design [**11**].

The rectilinear convex hull of point sets is an orientation-dependent region, i.e., it changes as the orientation of the plane changes. In this paper we are interested in computing an orientation for which the rectilinear convex hull of P has minimum area. We show that the set of orientations $\theta \in [0, 2\pi)$ can be divided into a set of linear intervals such that, within each interval I, the angle $\theta \in I$ which minimizes the area of the rectilinear convex hull of a point set (save the first one we process) can be calculated in constant time. These intervals can be computed in $O(n \log n)$ time and $O(n)$ space. Using this result and based on techniques from Avis et al. [**1**], Bae et al. [**2**], and Díaz-Báñez et al. [**4**], we present an optimal $\Theta(n \log n)$ time and $O(n)$ space algorithm for this problem. Our result improves the $O(n^2)$ time complexity presented by Bae et al. [**2**].

1 Terminology and notation

An *orthogonal wedge* is the intersection of two open half-planes whose supporting lines are orthogonal. The *apex* of the wedge is the intersection point of these supporting lines. An orthogonal wedge is *P-free* if it does not contain points of P in its interior. An orthogonal wedge is called a *θ-wedge* if its supporting lines can be obtained by first rotating the X-

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and Y-axis θ degrees, and then translating the origin to the apex of our wedge. The *rectilinear convex hull of P with orientation θ* is the region

$$\mathcal{RH}_\theta(P) = \mathbb{R}^2 - \bigcup_{w \in \mathcal{W}_\theta} w,$$

where $\mathcal{W}_\theta$ is the set of all P-free orthogonal θ-wedges [**2, 4, 8**].

As θ changes, the set of orthogonal P-free θ-wedges change, and, thus, $\mathcal{RH}_\theta(P)$ changes (see Figure 1). A θ-orientation of the plane, $\theta \in [0, 2\pi)$, is the coordinate system obtained by rotating the axes of $\mathbb{R}^2$ by θ degrees with respect to the origin. For a fixed θ, $\mathcal{RH}_\theta(P)$ has a close relation to the maxima problem [**6, 9**]. A *vertex* of $\mathcal{RH}_\theta(P)$ is a point in P that lies on the boundary of $\mathcal{RH}_\theta(P)$. Let $X_\theta(P)$ be the set of maximal points of P with respect to vector dominance in the θ-orientation of the plane. The set of vertices of $\mathcal{RH}_\theta(P)$ is equal to the set $X_\theta(P) \cup X_{\theta+\frac{\pi}{2}}(P) \cup X_{\theta+\pi}(P) \cup X_{\theta+\frac{3}{2}\pi}(P)$ [**2, 8**]. Given a fixed θ, $\mathcal{RH}_\theta(P)$ can be computed in optimal $\Theta(n \log n)$ time and $O(n)$ space [**6, 9**].

We say that a point $p \in P$ is *θ-maximal with respect to P* if there is an orthogonal P-free wedge with apex at p in a θ-orientation of the plane. The set of orientations for which p is θ-maximal forms at most three intervals. The endpoints of each interval mark the *in*- and an *out*- events of p, i.e., the θ-orientations when p becomes and stops being θ-maximal. The set of intervals corresponding to the elements of P and the set of angles at which these points of P start and stop being θ-maximal can be computed in optimal $\Theta(n \log n)$ time and $O(n)$ space [**4**].

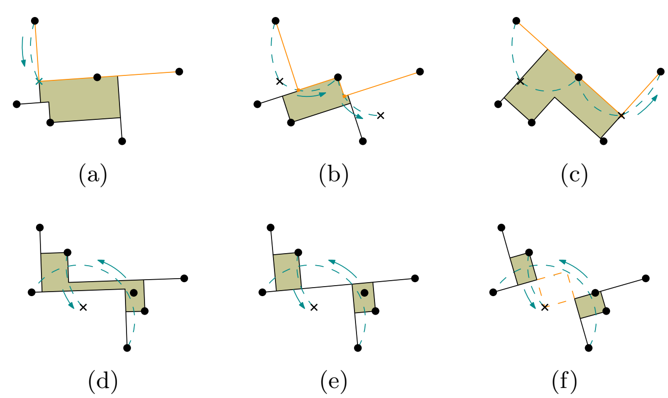


FIGURE 1. The rectilinear convex hull of P changes with the orientation.

Let *X_θ-axis* and *Y_θ-axis* denote the coordinate axes rotated θ degrees. For a θ orientation, consider the coordinates of the points of P in terms of the X_θ- and Y_θ-axes. Since $\mathcal{RH}_\theta(P)$ is monotone with respect to the X_θ-axis [**8**], the points of P can be re-labelled as $v_1, \ldots, v_m$ in increasing order according to X_θ. Two consecutive points $v_i, v_{i+1} \in P$ with respect to X_θ define the *step* $s_\theta(v_i, v_{i+1})$. Given two orientations α and β, we say that two steps $s_\alpha(v_i, v_{i+1})$ and $s_\beta(v_j, v_{j+1})$ are *opposite* to each other if $|\alpha - \beta| = \pi$; see Figure 1(b). Every step $s_\theta(v_i, v_{i+1})$ supports a P-free θ-wedge. Let W_1 and W_2 be the wedges supported by two opposite steps $s_\theta(v_i, v_{i+1})$ and $s_{\theta+\pi}(v_j, v_{j+1})$, respectively. If W_1 and W_2 intersect, $\mathcal{RH}_\theta(P)$ is disconnected. In such case, we say that $s_\theta(v_i, v_{i+1})$ and $s_{\theta+\pi}(v_j, v_{j+1})$ *overlap*, and denote $W_1 \cap W_2 = t_\theta(i, j)$; see Figure 1(f).

Consider four points v_i, v_{i+1} and v_j, v_{j+1} and let I be the the interval of orientations θ for which $s_\theta(v_i, v_{i+1})$ and $s_{\theta+\pi}(v_j, v_{j+1})$ overlap. As before, we call the ends of I the *start*- and *stop*-events of $t_\theta(i, j)$; see Figures 1(e) and 1(f). We wish to compute a counterclockwise ordered start- and stop-event list that resembles the one we computed for in- and out-events of the elements of P. Overlap events are not necessarily vertex events and thus, they have to be computed independently.

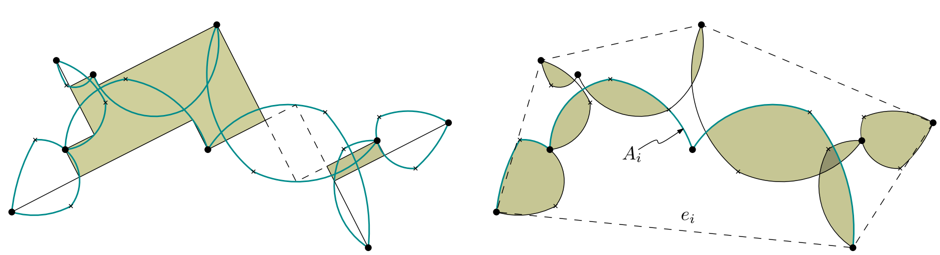
2 Computing the start- and stop-overlap events list

The *apex* of a step is the apex of the wedge that it supports. As θ changes from 0 to 2π, the θ-orientation of the plane *rotates* counterclockwise, and the apex of every step traces a circular arc. We orient the arcs traced by the elements of P as shown in Figures 1(a)–1(c). The *arc-chain* $\mathcal{A}(P)$ of P is the closed curve formed by the union of the set of arcs traced by the elements of P which, at some point in time are θ-maximal for some $\theta \in [0, 2\pi)$, let $\mathcal{A}(P) = \langle a_1, \ldots, a_l \rangle$ (Figure 2(a)). Since there is a linear number of steps in a complete rotation, $l = O(n)$. Observe that the endpoints of the arcs in $\mathcal{A}(P)$ include the points in P that are θ-maximal for some $\theta \in [0, 2\pi)$.

Let $\{e_1, \ldots, e_h\}$, be the set of edges of the convex hull $\mathcal{CH}(P)$ of P in counterclockwise order. A *sub-chain* $A_i(P)$ of $\mathcal{A}(P)$ is a subsequence of consecutive elements of $\mathcal{A}(P)$, whose endpoints are the endpoints of e_i. It is easy to see that $A_i(P)$ is monotone in the direction determined by e_i. Thus the orthogonal projection of $A_i(P)$ on e_i defines a total order $(\prec_i)$ on the set of endpoints of its arcs. Moreover, using the fact that every point in $A_i(P)$ is an apex of a P-free wedge, the next lemma follows easily.

Lemma 2.1. *Let a, b, c be three points in $A_i(P)$ such that $a \prec_i b \prec_i c$. Then, the angle $\angle abc$ is such that $\frac{\pi}{2} \leq \angle abc < \pi$.*

Suppose that we relabel the endpoints of the arcs in $A_i(P)$ as $p_1, \ldots, p_m$ so that, if $r < s$, then $p_r \prec_i p_s$. Let $\ell_{r,s}$ be a subsequence $p_r, \ldots, p_s$ of $A_i(P)$ such that for $r < t < s$, $p_t \notin P$ and $p_r, p_s \in P$. We call any such $\ell_{r,s}$ a *link*. Observe that, if two opposite steps overlap, then the arcs traced by their apices belong to links that intersect; see Figure 2(a). The open area bounded by $A_i(P)$ and e_i is P-free, since it is covered by P-free wedges. Thus, two intersecting links have at least two intersection points. By Lemma 2.1, this number is tight, as none of the intersecting links can cross a line segment joining its intersection points; see Figure 2(b). Thus we have the following result, that is a central tool for computing the start- and stop-overlap event list in $O(n \log n)$ time.



(a) (b)

FIGURE 2. The *arc-chain* $\mathcal{A}(P)$ of P.

Theorem 2.2. *There are $O(n)$ intersections between links in $\mathcal{A}(P)$.*

3 Computing the orientation of $\mathcal{RH}_\theta(P)$ with minimum area

The event points obtained in the previous section generate a set of intervals of orientations, in which the set of vertices of $\mathcal{RH}_\theta(P)$ remain unchanged, and the set of overlaps among the steps of $\mathcal{RH}_\theta(P)$ does not change. Let $I_{(\theta_1,\theta_2)}$ be one such interval. Then, for any $\theta \in (\theta_1, \theta_2)$, the area of $\mathcal{RH}_\theta(P)$ is given by the following formula [**2**]:

$$\operatorname{area}(\mathcal{RH}_\theta(P)) = \operatorname{area}(\mathcal{P}) - \sum \operatorname{area}(s_\theta(v_i, v_{i+1})) + \sum \operatorname{area}(t_\theta(i,j)).$$

It is easy to see that the areas of the steps $s_\theta(v_i, v_{i+1})$ and overlaps $t_\theta(i,j)$ of $\mathcal{RH}_\theta(P)$ can be expressed as a function of $\sin 2\theta$ and $\cos 2\theta$. Doing the derivative, we obtain:

$$\sum \operatorname{area}'(s_\theta(v_i, v_{i+1})) = -\big[\sum A_i\big] \sin 2\theta + \big[\sum B_i\big] \cos 2\theta, \tag{1}$$

$$\sum \operatorname{area}'(t_\theta(i,j)) = \big[\sum C_i\big] \cos 2\theta - \big[\sum D_i\big] \sin 2\theta, \tag{2}$$

and thus the value $\theta \in (\theta_1, \theta_2)$ for which the area of $\mathcal{RH}_\theta(P)$ is minimized can be computed in linear time. For each new event interval, we update these values in constant time by subtracting or adding new constant values. There can be more than one θ-orientation in which $\mathcal{RH}_\theta(P)$ has minimum area, but our algorithm is able to report all of them. From the discussion above and from the fact that the convex hull of P can be computed from the rectilinear convex hull of P in $O(n)$ time, we obtain the following:

Theorem 3.1. *Computing the set of orientations for which the rectilinear convex hull of P has minimum area can be done in optimal $\Theta(n \log n)$ time and $O(n)$ space.*

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