Separation of convex sets

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Abstract

A line L separates a set A from a collection S of plane sets if A is contained in one of the closed half-planes defined by L, while every set in S is contained in the complementary closed half-plane. Let $f(n)$ be the largest integer such that for any collection F of n closed disks in the plane with pairwise disjoint interiors, there is a line that separates a disk in F from a subcollection of F with at least $f(n)$ disks. In this note we prove that there is a constant c such that $f(n) > \frac{(n-c)}{2}$. An analogous result for the d-dimensional Euclidean space is also discussed.

A line L separates a set A from a collection S of plane sets if A is contained in one of the closed half-planes defined by L, while every set in S is contained in the complementary closed half-plane.

Alon et al. proved in [1] that there is a constant $a > 0$ such that, for any collection F of n congruent disks in the plane with pairwise disjoint interiors, there is a line L that leaves at least $\frac{n}{2} - a\sqrt{(n \ln n)}$ disks of F on each closed half-plane defined by L.

We denote by $f(n)$ the largest integer such that for any collection F of n closed disks in the plane with pairwise disjoint interiors and arbitrary radii, there is a line that separates a disk in F from a subcollection of F with at least $f(n)$ disks. Czyzowicz et al. proved in [2] that $\frac{n}{2} \geq f(n) \geq \frac{(n-7)}{4}$. In this note we prove that there is a constant c such that $f(n) \geq \frac{(n-c)}{2}$.

Let A be a compact convex set in the plane with nonempty interior, we denote by $e(A)$ the ratio $\frac{D(A)}{r(A)}$, where $D(A)$ is the diameter of A and $r(A)$ is the radius of the largest disk inscribed in A. We prove the following stronger result:

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Theorem 1 *For each positive real number e, there is a constact $c(e)$ such that if F is a collection of n compact convex sets in the plane with pairwise disjoint interiors and such that $e(A) \leq e$ for every set A in F, then there is a line that separates a set in F from a subcollection of F with at least $\frac{(n-c(e))}{2}$ sets.*

In order to prove Theorem 1 we establish some notation and a lemma. If S and T are compact sets in the plane, we denote by $d(S,T)$ the distance between S and T; that is $d(S,T) = min\{d(s,t) : s \in S, t \in T\}$. For any nonnegative real number r, we denote by B_r the disk with radius r centered at the origin.

Lemma 1 *For each positive real number e, there is a constant $c(e)$ such that if F is a collection of $c(e)+1$ or more compact convex sets in the plane with pairwise disjoint interiors and such that $e(A) \leq e$ for every set A in F, then there are two sets S and T in F such that $d(S,T) > D$ where D is the smallest diameter among the sets in F.*

Proof: Let e be a positive real number and let $F = \{K_1, K_2, \ldots, K_m\}$ be a collection of m compact convex sets in the plane with pairwise disjoint interiors such that $e(A) \leq e$ for every set A in F. Without loss of generality we assume that K_1 is the set in F with the smallest diameter $D = D(K_1)$ and consider the set P given by the Minkowski sum $K_1 + B_D = \{x + y : x \in K_1, y \in B_D\}$. Suppose $d(K_i, K_j) \leq D$ for every pair of sets in F; we shall prove that m is bounded by a constant that depends only on e.

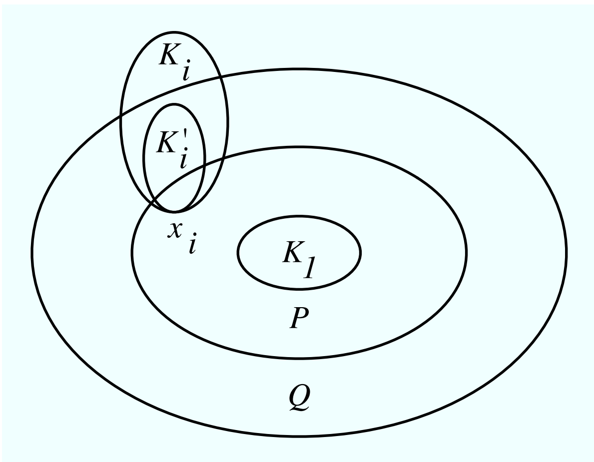

Figure 1:

Since $d(K_1, K_j) \leq D$ then every set in F intersects P. For $i = 2, 3, \ldots, m$, let x_i be any point in $K_i \cap P$ and let $K_i^{'} = (1 - t_i)x_i + tK_i$, where $t_i = \frac{D}{D(K_i)}$. Notice that $K_i^{'}$ has diameter D and $x_i \in K_i^{'}$, therefore $K_i^{'}$ is contained in the set $Q = P + B_D$ (see Fig. 1).

Let $K_1^{'} = K_1$; then for $i = 1, 2, \ldots, m$, $K_i^{'}$ is homothetic to K_i and therefore $e(K_i^{'}) = e(K_i) \leq e$. Since $D(K_i^{'}) = D$ then $r(K_i^{'}) \geq \frac{D}{e}$ and then $K_i^{'}$ contains a disk of radius $\frac{D}{e}$. Since $K_i^{'}$ is contained in Q for $i = 1, 2, \ldots, m$, then Q contains m disks with radius $\frac{D}{e}$ and pairwise disjoint interiors. Notice that $D(Q) = 5D$ and that the area of Q is at most $25D^2$, therefore $m\pi(\frac{D}{e})^2 \leq 25D^2$ and then $m \leq \frac{(25e^2)}{\pi}$. ■

Proof of Theorem 1. Let F be collection of n compact sets in the plane with pairwise disjoint interiors such that $e(A) \leq e$ for every set A in F. Let C_1 and C_2 be a pair of sets in F for which the distance $d = d(C_1, C_2)$ is maximum. Let L_i and L_2 be the parallel lines at distance d that separate C_1 and C_2, (see Fig. 2).

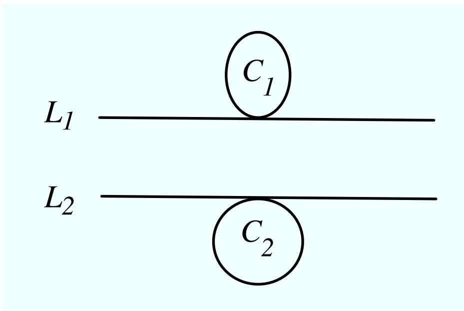

Figure 2:

Without loss of generality assume L_1 and L_2 are horizontal with L_1 above L_2. Every set in F is in at least one of the following subcollections: X_1 consists of those sets in F that lie in the closed half-plane below L_1, X_2 consists of those sets in F contained in the closed half-plane above L_2 and X_3 consists of those sets in F that intersect both lines L_1 and L_2. Notice that L_1 separates C_1 from all sets in X_1 and L_2 separates C_2 from all sets in X_2.

Since L_1 and L_2 meet every set in X_3, then the smallest diameter among the sets in X_3 is at least d; by Lemma 2 and the choice of C_1 and C_2, the class X_3 contains at most $c(e)$ sets, therefore $X_1 \cup X_2$ contains at least $n - c(e)$ sets and at least one of the collections X_1 or X_2 contains at least $\frac{(n-c(e))}{2}$ sets. ■

An immediate consequence of Theorem 1 is the following result:

Corollary 1 *There is a constant c such that $f(n) \geq \frac{(n-c)}{2}$.*

For a compact convex set A in the d-dimensional Euclidean space E^d, we denote by $e_d(A)$ the ratio $\frac{D(A)}{r_d(A)}$, where $D(A)$ is the diameter of A and $r_d(A)$ is the radius of the largest d-dimensional ball contained in A.

The following result may be proved with arguments analogous to those of Lemma 1 and Theorem 1.

Theorem 2 *For each positive real number* e *and every positive integer* d*, there is a constant* $c(e, d)$ *such that if* F *is a collection of* n *compact sets in* E^d *with pairwise disjoint interiors and such that* $e_d(A) \leq e$ *for every set* A *in* F*, then there is a hyperplane that separates a set in* F *from a subcollection of* F *with at least* $\frac{(n-c(e,d))}{2}$ *sets.*

References

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[2] J. Czyzowicz, E. Rivera-Campa, J. Urrutia and J. Zaks, Separating convex sets in the plane, Discrete Comput. Geom. 7 (1992) 189-195.