Flipping Edges in Triangulations of Point Sets, Polygons and Maximal Planar Graphs

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Abstract

A triangulation of a point set P_n is a partitioning of the convex hull $Conv(P_n)$ into a set of triangles with disjoint interiors such that the vertices of these triangles are in P_n, and no element of P_n lies in the interior of any of these triangles. An edge e of a triangulation T is called flippable if it is contained in the boundary of two triangles of T, and the union of these triangles forms a convex quadrilateral C. By flipping e we mean the operation of deleting e from T and replacing it by the other diagonal of C. A triangulation of a polygon Q_n is a partition of Q_n into a set of $n-2$ triangles with disjoint interiors such that the edges of these triangles are vertices of Q_n.

In this paper we will prove that any triangulation of a point set (polygon) can be transformed into any other by a sequence of flips. We will also prove that there are triangulations of point sets (polygons) such that to transform one into the other takes $O(n^2)$ flips. We prove that any triangulation of any set of points contains at least $\lfloor \frac{n-4}{2} \rfloor$ flippable edges. Motivated by this result, we generalize the concept of flipping edges to that of simultaneously flipping sets of independent

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edges, i.e. edges such that the quadrilaterals formed by the union of the triangles containing them are quadrilaterals with disjoint interiors. We show that for parallel flips, the flipping distance between triangulations of convex point sets is $O(n \log n)$.

We will also study the problem of flipping edges in *labelled triangulated graphs*, i.e. labelled planar graphs with exactly $3n-6$ edges. A classical result of Wagner asserts that any *unlabelled* triangulated graph can be transformed into any other by a sequence of edge flips. We will prove that the flipping distance for labelled planar triangulations is at most $O(n \log n)$. We also prove that any planar triangulation contains at least $n-2$ flippable edges. This bound is tight.

1 Introduction

Let $P_n = \{v_1, \ldots, v_n\}$ be a collection of points on the plane. A *triangulation of* P_n is a partitioning of the convex hull $Conv(P_n)$ of P_n into a set of triangles $T = \{t_1, \ldots, t_m\}$ with disjoint interiors such that the vertices of each triangle t_i of T are points of P_n. The elements of P_n will be called the vertices of T and the edges of the triangles $t_1, \ldots, t_m$ of T will be called the edges of T. The degree $d(v_i)$ of a vertex v_i of T is the number of edges of T that have v_i as an endpoint. We say that an edge e of T is *flippable* if e is contained in the boundary of two triangles t_i and t_j of T such that $C = t_i \cup t_j$ is a convex quadrilateral. By *flipping* e we mean the operation of removing e from T and replacing it by the other diagonal of C. See Figure 1.

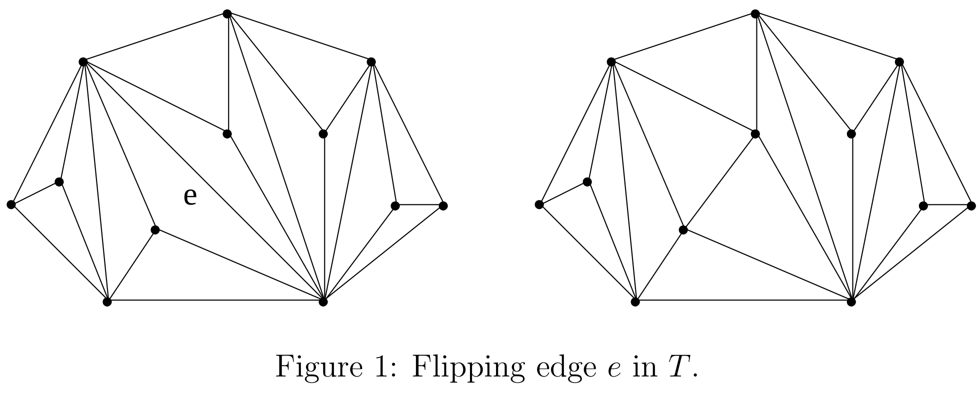

Figure 1: Flipping edge e in T.

Given a collection of points P_n we define the graph of triangulations $G_t(P_n)$ to be the graph whose vertex set is the set of triangulations of P_n;

two triangulations of $G_t(P_n)$ being adjacent if one can be obtained from the other by an edge flip.

Given two triangulations T' and T'' of P_n we say that they are at distance k if as vertices of $G_t(P_n)$ they are at distance k. In this case we will say that T' can be transformed into T by flipping k edges. Triangulations of polygons with or without holes, and the operation of flipping edges in them are defined in an analogous way.

Triangulations of point sets and polygons on the plane have been studied intensely in the literature both because of their intrinsic beauty and for their use in many problems, such as image processing [38], mesh generation for finite element methods [3, 16, 39, 46], scattered data interpolation [27, 34] and many others such as computer graphics, solid modeling and geographical information systems [1, 4, 10, 33, 35, 36, 37, 41, 44, 45].

It is well known that if P_n is convex, i.e. the vertex set of a convex polygon, then the diameter of $G_t(P_n)$ is at most $2n-3$. Graphs of triangulations of convex sets of points have been studied in [15, 40]. If P_n is convex, $G_t(P_n)$ is isomorphic to the rotation graph $RG(n-2)$. The vertex set of $RG(n-2)$ is the set of all binary trees with $n-2$ vertices [40].

It is known that any triangulation can be transformed to the Delauney triangulation by a simple greedy algorithm (see [12]).

It is also known that the graph of triangulations of a simple polygon Q_n with n vertices is connected [4, 12, 20, 24, 25, 33] and that its diameter is at most $O(n^2)$ [15].

Some additional results on the graph of triangulations of convex polygons have been obtained in [15].

In Section 2 we give a new and simple proof that the graph of triangulations of a polygon Q_n with n vertices with or without holes is connected. Next we show that there are polygons with n vertices such that the diameter of their graph of triangulations is $O(n^2)$. We then develop two algorithms that transform any triangulation T of Q_n into any other triangulation T''. The number of flips required by our first algorithm is at most the number of edges of the visibility graph of Q_n. Our second algorithm uses at most $O(n+k^2)$ flips where k is the number of reflex vertices of Q_n.

In Section 3 we study triangulations of point sets on the plane. Our main result in that section is to prove that any triangulation of a point set P_n with n points on the plane contains at least $\lfloor \frac{n-4}{2} \rfloor$ flippable edges. Our bound is tight.

Graph theorists have also studied flipping edges in maximal planar graphs, i.e. planar graphs with exactly $3n-6$ edges. To this end, consider a maximal planar graph G and an embedding of it on the plane. This embedding induces a *topological* partition of the plane into a set of *triangular regions*. In this sense, a triangular region is one bounded by three edges of G. These regions need not be convex, and the edges of G need not be line segments as we required before. The edges bounding a face of an embedding of G will be called the faces of T. That the faces of T are well defined follows from a well known result that up to isomorphisms maximal planar graphs have a unique embedding on the plane.

This allows us to define the concept of flipping edges on planar triangulations as follows: let v_iv_j be an edge of a planar triangulation T, and $\{v_i, v_j, v_k\}$ and $\{v_i, v_j, v_l\}$ be the vertices of the faces of G containing v_iv_j on their boundaries. We say that v_iv_j is flippable if v_k and v_l are not adjacent in T. By flipping v_iv_j, we mean the operation of removing it from T followed by the insertion of v_kv_l into T.

It is easy to see that this produces a new graph T' which is also a planar triangulation; see Figure 2. This operation is called a *diagonal flip* on ab, and ab is called a flippable edge in T.

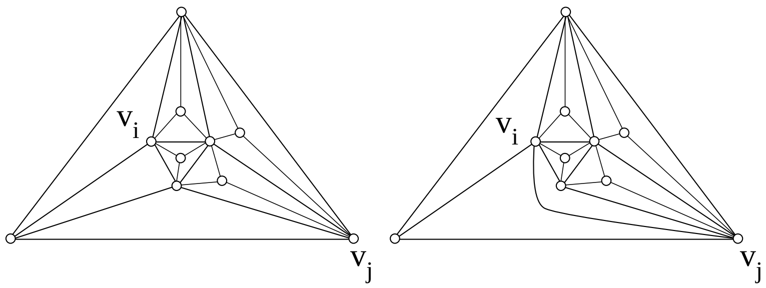

Figure 2: Flipping an edge in a planar triangulation

A classical result of Wagner states that any two planar *unlabelled* triangulations with the same number of vertices can be transformed into each other by a sequence of diagonal flips.

Let $\mathcal{T}_n$ be the set of all planar triangulations with n vertices. The *diagonal flip adjacency graph* denoted by $G_{\mathcal{T}}$ is the graph with vertex set $\mathcal{T}_n$, two members of $\mathcal{T}_n$ being adjacent if and only if one can be transformed into other by a single diagonal flip. In this language Wagner's result implies that $G_{\mathcal{T}}$ is connected.

Dewdney [7], Negami and Watanabe [32] have shown similar results for triangulations of the torus, the projective plane and the Klein bottle. It is easy to see that Wagner's result extends to labelled planar triangulations. However it is not always true for labelled triangulations on the projective plane, the torus and the Klein bottle, since there are different triangular embeddings of a labelled complete graph in each of these surfaces. For triangulations in general surfaces, $G_{\mathcal{T}}$ need not be connected even for unlabelled triangulations [29]. However, Negami [31] showed that for any surface Σ, there is a constant L such that $G^{l}_{\mathcal{T}}$ is connected for labelled triangulations with at least L vertices. Very recently Komuro, Nakamoto and Negami [22, 30] obtained similar results for triangulations with minimum vertex degree at least 4. Diagonal flips preserving some specified properties are discussed in [5].

We point out here that Wagner's original argument for unlabelled planar triangulations gives a quadratic bound on the diameter of $G_{\mathcal{T}}$. That the diameter of $G_{\mathcal{T}}$ is linear was recently proved by Komuro [21]. The difference in the diameter of $G_{\mathcal{T}}$ and the diameter of graph of triangulations of point sets leads in a natural way to the study of $G_{\mathcal{T}}$ for labelled graphs: the positions of the elements of a point set make them *labelled* points in a natural way. To be more precise, in the last section of this paper we study the following problem. Let $V = \{v_1, \ldots, v_n\}$ be a set of vertices, and let G and G' be two planar triangulations with vertex set V. How many edge flips are needed to transform G into G'? We will prove that the flipping distance for labelled planar triangulations is at most $O(n \log n)$. We also prove that any planar triangulation contains at least $n-2$ flippable edges. This bound is tight. In Figure 18 we show two labelled triangulations on $\{v_1, \ldots, v_6\}$ such that to transform one into the other requires 2 flips. Notice that as unlabelled triangulations, the triangulations shown in the same figure are isomorphic, and no flipping is needed to transform one into the other; however as labelled triangulations they are different.

2 Flipping edges in polygons

Let Q_n be a simple polygon with n vertices. Assume that the vertices of Q_n are labelled $v_1, \ldots, v_n$ in the clockwise direction around its boundary. The *visibility graph* of Q_n is the graph with vertex set $\{v_1, \ldots, v_n\}$. Two vertices v_i and v_j of Q_n are adjacent in the visibility graph of Q_n if the line segment joining them is contained in Q_n. We now prove:

Theorem 1 *The graph of triangulations $G_t(Q_n)$ of a simple polygon is connected. Moreover, the diameter of $G_t(Q_n)$ is proportional to the number of edges of the visibility graph of Q_n.*

Some definitions and preliminary results will be needed to prove our result.

Let T be a triangulation of Q_n, and let v_i, v_j be non-adjacent vertices in T. We say that v_iv_j can be inserted in T by flipping k edges if there is a sequence of triangulations $T_1 = T, \ldots, T_k$ such that v_iv_j is an edge in T_k and T_{i+1} can be obtained from T_i by flipping one of its edges, $i = 1, \ldots, k-1$. We say that a vertex v_i of Q_n is *exposed* if it lies in the convex hull of Q_n. Consider the two vertices v_{i-1} and v_{i+1} of Q_n adjacent to v_i. The shortest polygonal chain joining v_{i-1} to v_{i+1} totally contained in Q_n will be denoted by $P_{i-1,i+1}$.

Lemma 1 *Let v_i be an exposed vertex of Q_n and T a triangulation of Q_n. Then it is always possible to insert $P_{i-1,i+1}$ into T using exactly as many flips as the number of edges of T, not in $P_{i-1,i+1}$, that intersect it.*

Proof: Let $w(P_{i-1,i+1})$ be the number of edges in T that cross it. Suppose that $w(P_{i-1,i+1}) > 0$ and let v_iv_k be the longest edge in T that crosses an edge in $P_{i-1,i+1}$. It is easy to see that v_iv_k can always be flipped, decreasing $w(P_{i-1,i+1})$ by one; see Figure 3. ∎

We now proceed to prove Theorem 1:

Proof: Let v_i be an exposed vertex of Q_n, and T_1, T_2 two triangulations of Q_n. By Lemma 1 we can insert $P_{i-1,i+1}$ in T_1 and T_2 to obtain new triangulations T_1' and T_2' of Q_n. Delete from Q_n the subpolygon bounded by the vertices of $P_{i-1,i+1}$ and v_i. This will result in a collection of simple polygons with disjoint interiors. Each of these polygons has two triangulations induced by T_1' and T_2' respectively and fewer vertices than Q_n. Our result now follows by induction on the number of vertices of Q_n.

To prove that the diameter of $G_t(Q_n)$ is at most the size of its visibility graph, we notice that each edge incident with v_i is used at most twice while transforming T_1 into T_2, once while inserting $P_{i-1,i+1}$ into T_1, and a second time while inserting $P_{i-1,i+1}$ into T_2. ∎

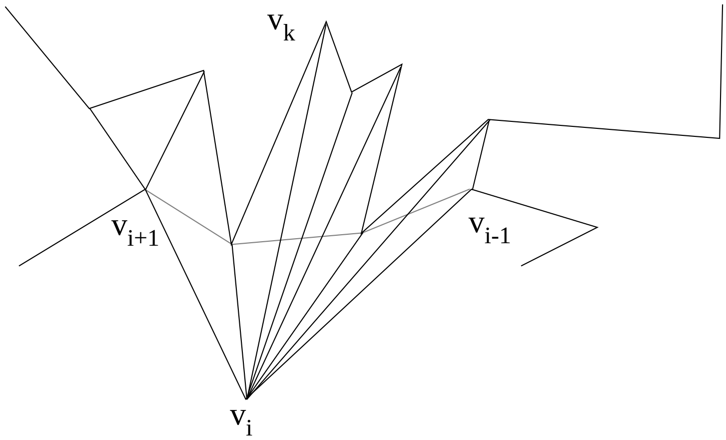

Figure 3: Inserting $P_{i-1,i+1}$ in T.

Consider the polygon R_n with $2n$ vertices $\{p_1, \ldots, p_n, q_1, \ldots, q_n\}$ such that:

a) $\{v_1, \ldots, p_n\}$ lie on a convex curve, and $\{q_1, \ldots, q_n\}$ lie on a concave curve.

b) The line joining $p_i$to p_j leaves all the elements of $\{q_1, \ldots, q_n\}$ below it, and all the elements of $\{p_1, \ldots, p_n\}$ lie above any line joining $q_i$to q_j, $1 \leq i < j \leq n$; see Figure 4.

We now show that there are two triangulations of R_n such that to transform one into the other requires exactly $(n-1)^2$ flips. This will prove our result. Consider any triangulation T of R_n. We assign a code to it as follows. Each triangle t_i of T has either two vertices in $\{p_1, \ldots, p_n\}$ or two vertices in $\{q_1, \ldots, q_n\}$. In the first case, assign a 1 to t_i; in the second case, t_i is assigned a 0 (see Figure 4).

If we read the numbers assigned to the triangles of T from left to right, we obtain an ordered sequence of 0's and 1's; this sequence is the code assigned to T. The triangulation presented in Figure 4 receives the code 01011100. It is clear that each triangulation of T is thus assigned a sequence containing exactly $n-1$ 0's and $n-1$ 1's. Clearly, each sequence of $n-1$ 0's and

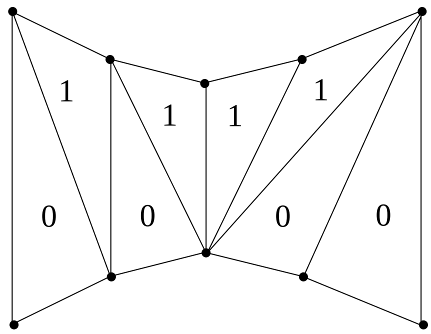

Figure 4: R_n, and a triangulation with code 01011100.

$n-1$ 1's also defines a unique triangulation of R_n. Thus we have a one-to-one correspondence between the set of triangulations of R_n and the set of binary sequences containing $n-1$ 0's and $n-1$ 1's. Flippable edges in these triangulations can be easily identified within this encoding. An internal edge of a triangulation T can be flipped if the triangles of T containing it have been assigned a 1 and a 0. Moreover, flipping an edge in T corresponds to a transposition of a 0 with a 1 in the code of T.

Consider the triangulations T_1 and T_2 of R_n that receive the encodings $11\ldots100\ldots0$ and $00\ldots011\ldots1$. It is now clear that to transform one into the other requires exctly $(n-1)^2$ flips, i.e. we have:

Theorem 2 *The diameter of $G_t(R_n)$ is $(n-1)^2$.*

Recall that Theorem 1 states that the diameter of $G_t(Q_n)$ is bounded by the size of the visibility graph of Q_n. However if Q_n is convex, the diameter of $G_t(Q_n)$ is lineal. It is thus natural to ask if there is some polygon parameter that allows us to obtain a better bound on the size of the diameter of $G_t(Q_n)$. The next result provides a partial answer to this question. We now prove:

Theorem 3 *Let Q_n be a simple polygon with k reflex vertices. Then the diameter of $G_t(Q_n)$ is at most $O(n+k^2)$.*

Two vertices v_i and v_j of a polygon Q_n are called c-connected if they are visible and the vertices $v_{i+1},\ldots,v_{j-1}$ of Q_n are all convex, addition taken mod n. If in addition, v_i and v_j are reflex vertices, we call them consecutive reflex vertices.

The segment v_iv_j will be called normal if either for each edge e of T intersecting it, the end vertex of e below v_iv_j is a convex vertex of Q_n, or for each edge e of T intersecting v_iv_j, the end vertex of e above it is a convex vertex of Q_n; see Fig 5.

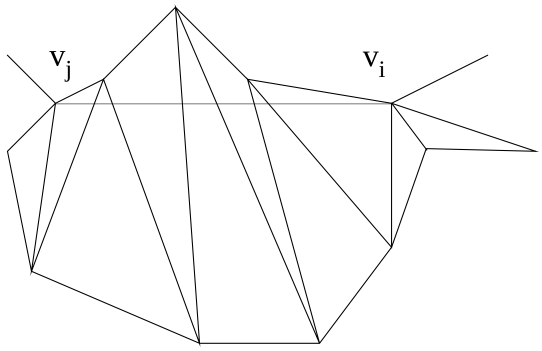

Figure 5: A normal diagonal of a triangulation.

The following lemma given without proof will prove useful to us. The proof of this result is similar to that of Lemma 1.

Lemma 2 *Let v_iv_j be a proper diagonal of a triangulation T of Q_n. Then if v_iv_j is intersected by k edges of T, we can insert it into T using at most $2t$ flips.*

A polygon Q is called spiral if the vertices of Q can be labeled $v_1, \ldots, v_s, v_{s+1}, \ldots, v_n$ such that $v_1, \ldots, v_s$ are reflex vertices and $v_{s+1}, \ldots, v_n$ are convex. The *standard* triangulation of a spiral polygon Q is now defined as follows. Let $p_{\sigma(1)}$ and $q_{\beta(1)}$ be the last reflex and convex vertex of Q visible from v_{n-1}. Join $p_{\sigma(1)}$ and $q_{\beta(1)}$. Next join v_{n-1} to all convex and reflex vertices of Q visible to it. Remove from Q v_n and all vertices visible from it, except $p_{\sigma(1)}$ and $q_{\beta(1)}$. This defines another spiral ploygon Q'. Iterate this process until we obtan a triangulation of Q; see Figure 6.

The following result is easy to prove:

Lemma 3 *Any triangulation of a spiral polygon Q_n can be transformed into the standard triangulation of Q_n with a linear number of flips.*

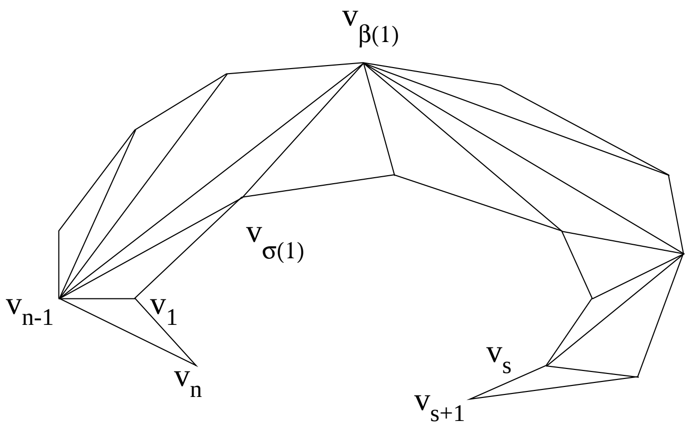

Figure 6: The standard triangulation of a spiral polygon.

Next suppose that Q_n has k reflex vertices labelled $v_{i_1}, \ldots, v_{i_k}$ such that $i_1 < \ldots < i_k$. For each $j = 1, \ldots, k$ let R_j be the shortest polygonal chain contained in Q_n joining v_{i_j} to $v_{i_{j+1}}$, addition taken mod k. Finally let $R = R_1 \cup \ldots \cup R_k$; see Figure 7.

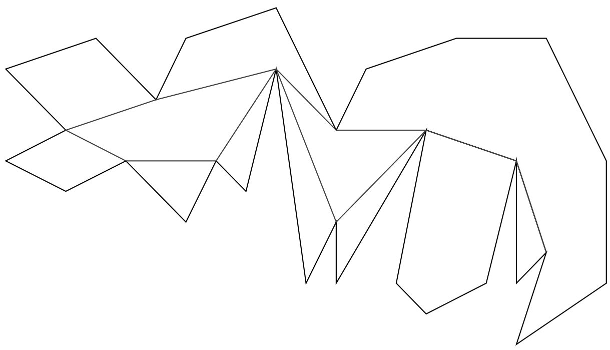

Figure 7: Constructing R.

The following lemma is easy to prove, and is given without proof:

Lemma 4 *Any edge joining two vertices of Q_n intersects at most two edges of R. Moreover if e is an edge of R and T is any triangulation of Q_n, e is either an edge of T or a proper diagonal of T.*

We now prove the last lemma needed to prove Theorem 3, namely:

Lemma 5 *Let T be any triangulation of Q_n. Then all the edges of R can be inserted into T using $O(n)$ flips.*

Proof: By Lemma 4, any edge of T intersects at most two edges of R. Since T has $n-3$ edges, the number of intersections between the edges of T and those of R is at most $2(n-3)$. However since all the edges of R are proper edges of T, each of these intersections can be removed by flipping at most two edges. Thus by flipping at most $4(n-3)$ edges, we can insert all the edges of R into T. ∎

We can now prove Theorem 3

Proof: Let T and T^* be triangulations of Q_n. By Lemma 4 we can insert the edges of R into each of them, flipping at most $4(n-3)$ edges and obtaining two new triangulations T_1 and T_1^* respectively. Notice that R induces a partitioning of Q_n into a set of polygons of one of these types:

a) At most k convex or spiral polygons $Q^1, \ldots, Q^m$, $k \leq m$ bounded by edges of Q_n or R.

b) A set of polygons $R_1, \ldots, R_s$ bounded by edges in R.

Notice that the total number of edges bounding $Q^1, \ldots, Q^m$ is at most $n+k$. Both T_1 and T_1^* induce possibly different triangulations in $Q^1, \ldots, Q^m$. Since $Q^1, \ldots, Q^m$ are convex or spiral polygons, all these triangulations can be transformed into each other by flipping $O(n+k)$ edges.

To end our proof, we notice that $R_1, \ldots, R_s$ are bounded by k edges, and thus the triangulations induced in them by T and T^* can be transformed into each other flipping $O(k^2)$ edges. Our result follows. ∎

3 Flipping edges in triangulations of point sets

We turn our attention to triangulations of point sets. The first thing to notice is that the proofs of Theorems 1 and 2 can be easily adapted to obtain equivalent results for triangulations of point sets. Thus we have:

Theorem 4 *The graph of triangulations of a point set is connected. Moreover, there are triangulations of the set of vertices of R_n such that to transform one into the other takes $O(n^2)$ flips.*

We now study the following problem for triangulations of point sets. An alert reader will notice that all triangulations of point sets have many flippable edges. It is thus natural to seek an answer to the following question:

> How many edges can be flipped in any triangulation of a point set?

We show:

Theorem 5 *Any triangulation of a set of n points in general position has at least $\lfloor \frac{n-4}{2} \rfloor$ flippable edges. Our bound is tight.*

In what follows, $Conv(P_n)$ denote the convex hull of a point set. Let T be a triangulation of a point set P_n. Divide the edges of T into two subsets, $F(T)$ the set of flippable edges of T, and $NF(T)$, the set of non-flippable edges in T.

Orient the edges of $NF(T)$ as follows:

a) Orient all the edges in $Conv(P_n)$ in the clockwise direction.

b) Let v_iv_j be an edge of T not in $Conv(P_n)$ and t_1 and t_2 be the triangles of T sharing v_iv_j. Let $C = t_1 \cup t_2$. Since v_iv_j is non-flippable, it follows that one of its end-vertices, say v_i, is a reflex vertex of C. Orient v_iv_j from v_j to v_i; see Figure 8.

The in-degree $d^-(v_i)$ of v_i the number of edges in $NF(T)$ oriented into v_i. We now prove:

Lemma 6 *Let v_i be any vertex in T. Then $d^-(v_i) \leq 3$. Moreover if v_i is incident with at least 4 edges in T, $d^-(v_i)$ is at most 2.*

Proof: If v_i is in $Conv(P_n)$, then $d^-(v_i) = 1$. Suppose then that v_i belongs to the interior of $Conv(P_n)$. Two cases arise:

CASE 1: If $d(v_i) = 3$, none of the edges incident with v_i is flippable, and they are oriented into v_i, thus $d^-(v_i) = 3$.

CASE 2: If $d(v_i) > 3$ it is easy to verify that that no more than two non-flippable edges of T can be oriented into v_i.

∎

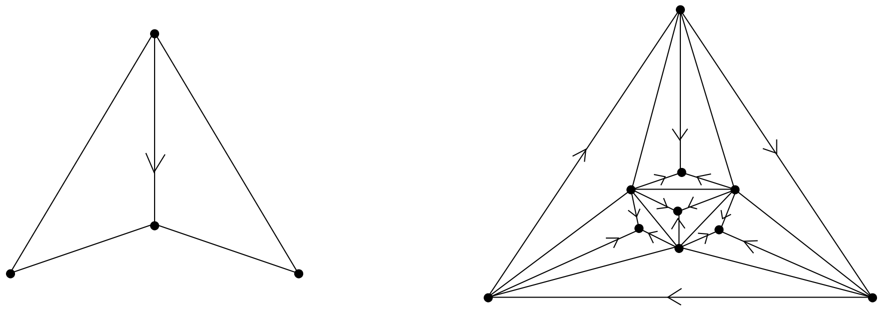

Figure 8: Orienting the non-flippable edges of a triangulation.

We prove now Theorem 5.

Proof: Let T be a triangulation of P_n, and S the set of vertices of T with degree 3 not in $Conv(P_n)$. By adding a point w in the exterior of P_n joined to the points of P_n on its convex hull by some edges, not necessarily represented by straight lines, we get a topological triangulation of the plane. Thus by Euler's theorem this triangulation contains $3n-3$ edges. Let us classify these new edges as non-flippable, and orient them towards $Conv(P_n)$. Orient all non-flippable edges according to the rules given above.

Notice that $d^-(v_i) = 2$ for all the vertices in the convex hull of P_n. Remove from T all the elements of S. Notice that we will remove exactly $3|S|$ edges of T which are not flippable. Furthermore, notice that what remains is still a triangulation T' of $P_n - S$, which by Euler's formula contains exactly $2(|P_n - S| + 1) - 4 \;=\; 2(n - |S|) + 2$ triangles. Moreover, any element v_i of T not in $Conv(P_n)$ has degree at least 4 in T, and thus by Lemma 6, $d^-(v_i) \leq 2$.

Let Q be the set of vertices v_i of $P - S$ that have $d^-(v_i) \;=\; 2$. Then by Lemma 6, we can associate to each element of Q a different triangle of T which is also a triangle in T. From the triangles having w as one of their vertices, we can also associate a different 'triangle' to each vertex of Q in $Conv(P_n)$. That is, to each vertex of Q, except w and the vertices of T with $d^-(v_i) < 2$, we can associate a different triangle of T' that contains no element of S.

Let m be the number of vertices of T that are on the boundary of $Conv(P_n)$ or have $d^-(v_i) \;=\; 2$. Since T' has $2(n - |S|) + 2$ triangles, it follows that $|S| \leq 2(n-|S|)+2-m$. It is easy to verify that the number of edges of T that can be flipped is minimized when all the vertices of $P_n - S$ have d^- equal to two.

In this case, since we can associate to each element of $P_n - S$ a different empty triangle of T', we can easily verify that $|S| = n - |S| - 2$, that is:

$$n = 2|S| + 2 \tag{1}$$

Since T' contains $3(|P_n - S| + 1) - 6 = 3(|P_n - S| - 3$ edges and each vertex of P_n has $d^-(v_i) = 2$, the number of flippable edges of T (i.e. those edges of T that are not oriented in T') is exactly :

$$k = (3(n - |S|) - 3) - 2(n - |S|) = n - |S| - 3 \tag{2}$$

Using (1) and (2) we get $k = \frac{n-4}{2}$ which concludes the first part of our proof.

We now show that our bound is tight. Take any collection of m points that are the vertices of a convex polygon Q_m, together with any triangulation of it. Next add to the interior of each triangle an extra vertex adjacent to the three vertices of the triangle. If the convex polygon has m vertices, our final point set has $2m - 2$ points, and the only edges that can be flipped are the $m - 3$ edges used to triangulate P_m. Trivially if $n = 2m - 2$, $m - 3 = \frac{n-4}{2}$; see Figure 9. ■

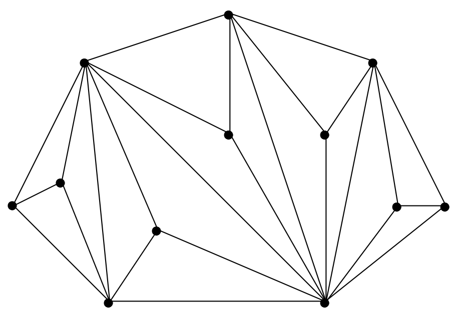

Figure 9: A triangulation of a point set with $2m-2$ points and $m-3$ flippable edges.

4 Flipping edges in triangulations of point sets in parallel

In the previous section, we showed that any triangulation of a point set contains a linear number of edges that can be flipped. This motivated us to study the following problem. Let S be a set of flippable edges of a triangulation of a point set. We say that S can be flipped simultaneously if no two elements of S bound a common triangle in T; see Figure 10. The operation of flipping all the elements of S at once will be called a *parallel flip*.

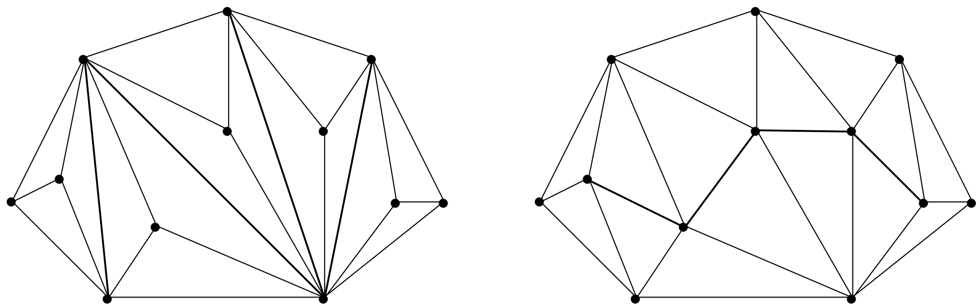

Figure 10: All the dark edges can be flipped simultaneously.

In this section we prove the following result:

Theorem 6 *Any two triangulations of a convex point set can be transformed into each other using at most $O(n \log n)$ parallel edge flippings.*

Before proceeding with our proof, we mention that a similar result exists for triangulations of arbitrary point sets, namely:

Theorem 7 ([18]) *Any two triangulations of a point set P_n can be transformed into each other using at most $O(n \log n)$ parallel flips.*

The proof of this result is involved; the interested reader can find the details in [18].

A triangulation of a convex point set P_n is called a *fan* triangulation if there is a point v of P_n adjacent to all the elements of P_n. The point v is called the *apex* of the triangulation. Notice that the dual graph of a triangulation T of a convex point set is a binary three H_T. The diameter of T is the maximum distance between two nodes of H_T. We now prove:

Lemma 7 *Let T be a triangulation of a convex point set P_n with diameter k, and v any element of P_n. Then T can be transformed to the fan triangulation of P_n with apex v using at most k parallel flips.*

Proof: Let T be a a triangulation of P_n, v any element of P_n, and e an edge of T not incident with v. The distance to v of an edge e is defined to be the number of edges of T intersected by any line segment joining any interior point of e to v plus 1; see Figure 11(a). It is now easy to see that at time i we can flip all the edges at distance i from v, $i = 1, \ldots, k$; see Figure 11. ∎

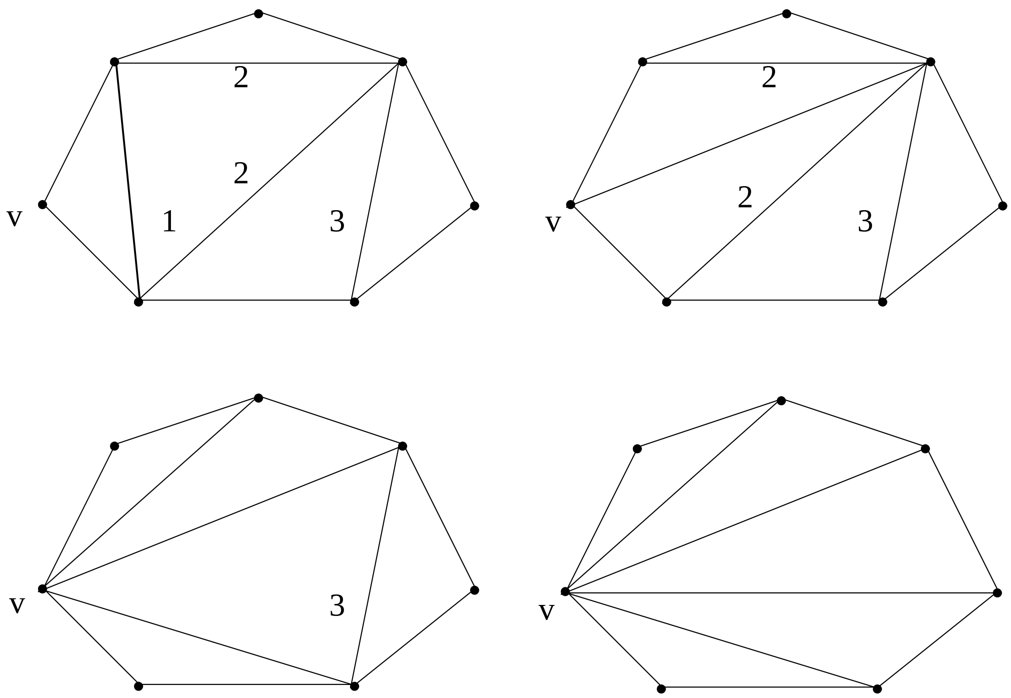

Figure 11: Transforming a triangulation to a fan triangulation.

Our problem is now that of transforming any triangulation of a convex point set to one with a logarithmic diameter in a logarithmic number of parallel flips.

Let H be a tree such that all the vertices of T have degree 3 or 1. Such a tree will be called a *leafy*tree. The following result justifies our terminology:

Lemma 8 *Any leafy tree has $2m+2$ vertices and $m+2$ leaves, $m > 0$.*

Let T be a triangulation of a convex point set. The dual graph of this triangulation is a tree H_T such that each vertex in it has degree 1, 2 or 3. If a face of T generates a leaf in H_T, it will be called a leaf of T. We now prove:

Lemma 9 *Any triangulation of a convex point set P_n can be transformed to one with at least $\lceil \frac{n+1}{5} \rceil + 2$ leaves using at most four parallel flips.*

Proof: We first observe that any tree obtained by subdividing any edge joining two non-leaf vertices of a leafy tree into at most 3 edges, and any edge joining a leaf to a nonleaf vertex at most once, has at least $\lceil \frac{n_1}{5} \rceil + 2$ leaves. We now show that using at most four parallel flips, any triangulation T of P_n can be transformed into a triangulation T' with at least $\lceil \frac{n+1}{5} \rceil + 2$ leaves.

Let v_1, v_2, v_3 be three vertices of degree 2 in the dual graph of a triangulation of a convex point set P_n. It is easy to see that by flipping at most two edges we can obtain a new triangulation of P_n with one more leaf; see Figure 12.

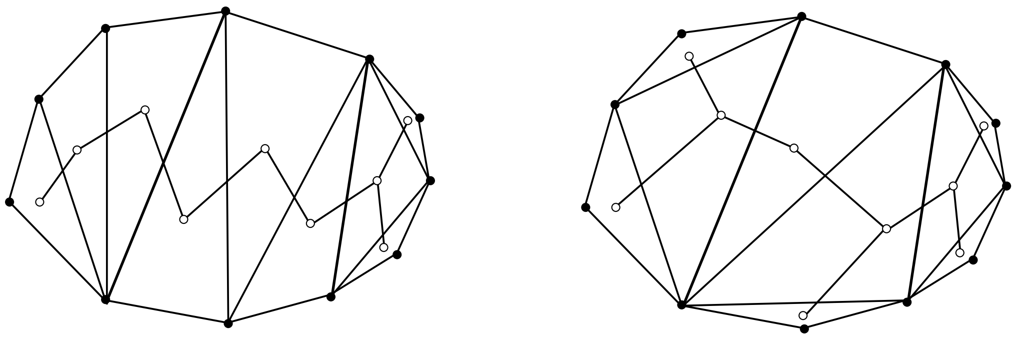

Figure 12: Eliminating long paths in the dual of a triangulation of a convex point set.

Consider the dual graph H_T of T. If it contains long paths consisting of vertices of degree 2 in H we can subdivide them into subpaths of length 3 plus one path of length 1 or 2. By the previous observation, using two parallel flips we can eliminate these paths and obtain a new triangulation

T' of P_n such that its dual graph $H_{T'}$ contains no induced paths of length greater than 3 consisting of vertices of degree 2. With two extra parallel flips we eliminate these paths, obtaining a triangulation T' such thatits dual graph contains at least $\lceil \frac{n+1}{5} \rceil + 2$ leaves. ∎

We now proceed to prove Theorem 6.

Proof: Let v be any element of P_n. We note that to prove our result, it is sufficient to show that any triangulation T of P_n can be transformed to the fan triangulation of P_n with apex v using a logarithmic number of parallel flips.

By using the procedure described in Lemma 9 we can transform T to a triangulation with at least $\lceil \frac{n+1}{5} \rceil + 2$ leaves. Let P' be the polygon obtained by joining the non-leaf faces of T'. By recursive applications of Lemma 9 to P' we obtain a triangulation H of P_n with logarithmic diameter.

By Lemma 7, using a logarithmic number of parallel flips, H can be transformed to the fan triangulation of P_n with apex v. Our result follows. ∎

5 Flipping edges in planar triangulations

We now turn our attention to the study of flipping edges in maximal planar graphs, i.e. in *planar triangulations*.

5.1 The diameter of G_T^l

Given two vertices v_i and v_j, a triangulation T will be called a $\Delta(i,j)$ triangulation if v_i and v_j are both adjacent to all vertices of T; see Figure 13. We will say that the edge joining v_i and v_j is the root edge of T. We now prove:

Theorem 8 *Let T be a triangulation and let v_i, v_j be adjacent vertices in T. Then we can transform T into a $\Delta(i,j)$ triangulation with at most $4n-16$ diagonal flips. Moreover let F be one of the triangular faces of T containing edge v_i v_j. Then $\Delta(i,j)$ can be chosen such that F is also a face in $\Delta(i,j)$.*

Proof: Let F be one of the two faces of T containing edge v_i v_j, and let v_k be the third vertex of F. We define the *potential* of T by

$$p_T(i,j) = 3\deg(v_i) + \deg(v_j),$$

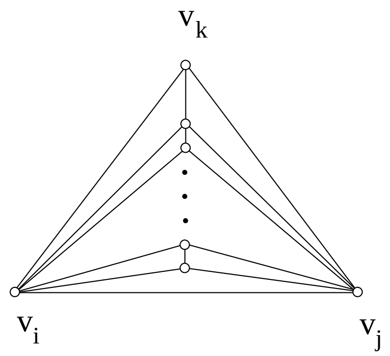

Figure 13: A $\Delta(i,j)$ triangulation.

and show that if T is not a $\Delta(i,j)$ triangulation, then by performing some diagonal flips we can increase $p_T(i,j)$. Note that $p_T(i,j) \leq 4(n-1)$ with the equality holding only when $\deg(v_i) = \deg(v_j) = n-1$, i.e. T is a $\Delta(i,j)$ triangulation.

Let $v_j, v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(i-1)}, v_{\sigma(l)} = v_k$ be the neighbors of v_i in T in the anticlockwise order. For convenience, we set $v_{\sigma(0)} = v_j$. Let m be the largest integer such that $v_{\sigma(1)}, v_{\sigma(2)}, \ldots v_{\sigma(m)}$ are all adjacent to v_k in T, and each triangle $v_j v_{\sigma(i-1)} v_{\sigma(i)}$ bounds a face for $i = 1, 2, \ldots, m$. If $m = l$, then T is a $\Delta(i,j)$ triangulation and no diagonal flip is needed. Otherwise let $v_j v_{\sigma(m)} u$ be the other face incident with the edge $v_j v_{\sigma(m)}$ with $u \neq v_{\sigma(m+1)}$. We distinguish the following two cases.

Case 1: u is not a neighbor of v_i. If $m = 1$ we can flip $v_j v_{\sigma(1)}$, and increase $p_T(i,j)$ by 2. If $m > 1$, we can flip $v_j v_{\sigma(m)}$ and then $v_{\sigma(m-1)} v_{\sigma(m)}$, and increase $p_T(i,j)$ by 2.

Case 2: $u = v_{\sigma(s)}$ for some $m+2 \leq s \leq l$. In this case $v_{\sigma(m)} v_{\sigma(s)}$ can be flipped and $p_T(i,j)$ increases by 1.

By iterating the above process, we transform T into a $\Delta(i,j)$ triangulation in which F remains a face. Notice that the total number of diagonal flips involved does not exceed $4(n-1) - p_T(i,j)$. Our result follows. ∎

Let T be a $\Delta(i,j)$ triangulation. Notice that $T - \{v_i, v_j\}$ is a path P. Assume without loss of generality that the vertices of P are labelled

$\{v_{\sigma(1)}, \ldots, v_{\sigma(n-2)}\}$. If the elements of P are such that $\sigma(1) < \sigma(2) < \ldots < \sigma(n-2)$ we say that T is *sorted.* A $\Delta(i,j)$ triangulation T' is called a *transpose* of T if $T' - \{v_i, v_j\}$ is a path P' obtained from P by transposing two consecutive vertices of P; see Figure 14.

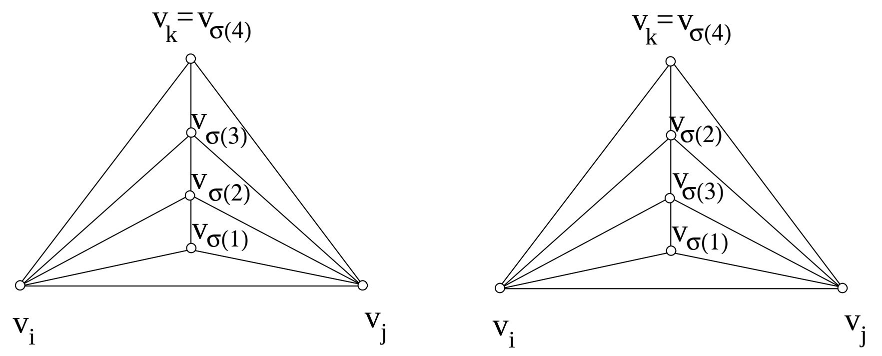

Figure 14: A $\Delta(i,j)$ triangulation and a transposition of it.

The next lemma is easy to prove:

Lemma 10 *Let T be a $\Delta(i,j)$ triangulation, and T' a transposition of T. Then T' can be obtained from T by flipping at most 4 edges.*

A $\Delta(1,2)$ triangulation such that the vertices of one of its faces are precisely v_1, v_2, v_n will be called a *normal* triangulation. We now prove:

Lemma 11 *Within $O(n)$ diagonal flips, any planar triangulation T with n vertices can be transformed into a normal $\Delta(1,2)$ planar triangulation T'.*

Proof: Let v_i be any neighbour of v_1. By Theorem 8 we can transform this triangulation into a $\Delta(1,i)$ triangulation using $O(n)$ diagonal flips. Since vertex v_2 is adjacent to v_1, again by Theorem 8, we can transform this triangulation into a $\Delta(1,2)$ triangulation with a linear number of flips. Using Lemma 10, we can now perform a linear number of transpositions until v_1, v_2 and v_n belong to the same face. ∎

Notice that using this lemma, together with Lemma 10, we can prove that any $\Delta(1,2)$ triangulation can be transformed to the sorted $\Delta(1,2)$ triangulation with a quadratic number of transpositions, i.e. a quadratic number of flips. We now proceed to show how to accomplish this in at most $O(n \log n)$ flips.

5.1.1 The binary triangulation

To achieve our goal, we define a special type of triangulation which we call *binary triangulations*. A planar triangulation with vertex set $\{v_1, \ldots, v_n\}$ is called binary if:

CASE 1: The vertices of a face of T are v_1, v_2, v_n.

CASE 2: The dual graph of $T - \{v_n\}$ (excluding the vertex corresponding to the only face of $T - \{v_n\}$ that is not a triangular face) is an almost balanced binary tree, i.e. any two paths starting at its root and ending at a leaf have the same length, or their lengths differ by at most one; see Figure 15.

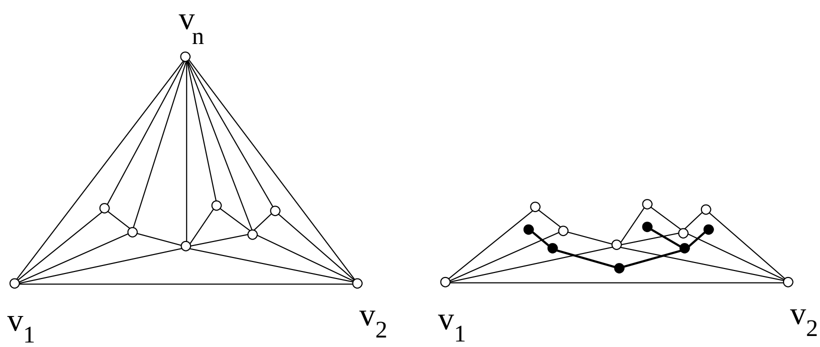

Figure 15: A binary triangulation.

We now proceed to show how a binary triangulation can be transformed to the sorted normal triangulation in at most $O(n \log n)$ flips.

A 2-Δ triangulation is a triangulation consisting of two vertex-disjoint sorted Δ triangulations $\Delta(i,j)$ and $\Delta(j,k)$ glued along an edge plus the edge joining v_i to v_k; see Figure 16(a).

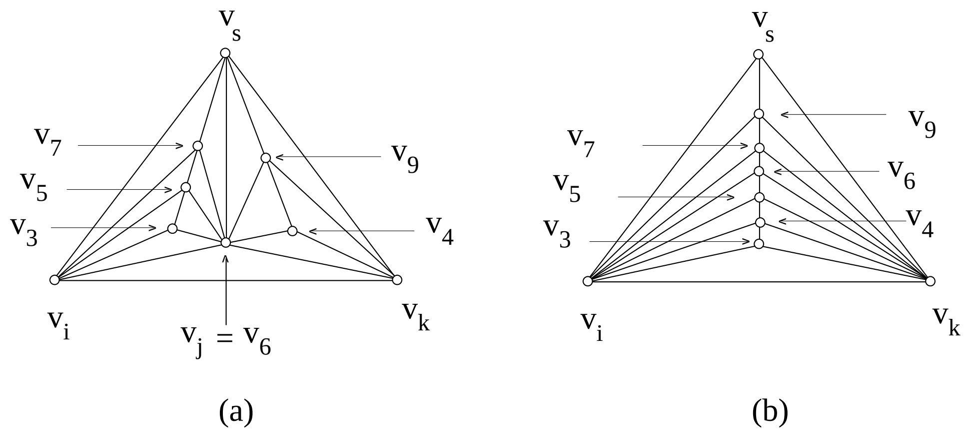

Figure 16: A 2-Δ triangulation and the resulting merged triangulation.

The following lemma will be essential to prove our main result:

Lemma 12 *Any 2-Δ triangulation can be transformed into a sorted Δ triangulation with a linear number of flips.*

Proof: Let $\Delta(i, j)$ and $\Delta(j, k)$ be the sorted Δ triangulations forming T, and let v_s be the common neighbour of v_i and v_k, as in Figure 16(a). Let $v_{\alpha(1)}, \ldots, v_{\alpha(r)}$ and $v_{\beta(1)}, \ldots, v_{\beta(t)}$ be the vertices in $\Delta(i, j)$ and $\Delta(j, k)$ respectively. Assume without loss of generality that $\alpha(1) < \beta(1)$. Then by performing two flips, we can obtain a triangulation in which $v_{\alpha(1)}$ is adjacent to v_i, v_j and v_k; see Figure 17(b). It is now easy to see that using three flips at a time, we can move the remaining vertices of $\Delta(i, j)$ and $\Delta(j, k)$ so that in the end we get an almost sorted $\Delta(i, k)$ triangulation. The only vertex out of place is perhaps v_j. This can be fixed by performing a linear number of transpositions until v_j moves to its correct position; see Figure 17(c),(d).$\blacksquare$

We are ready to prove:

Lemma 13 *Let T be a binary planar triangulation with n vertices. Then T can be transformed into the $\Delta(1, 2)$ sorted triangulation by performing $O(n \log n)$ diagonal flips.*

Proof: Our theorem is true for $2^2 \leq n \leq 2^3$. Suppose that the result is true for $2^{i-1} \leq n \leq 2^i$. We now show that it also holds for $2^i \leq n \leq 2^{i+1}$. Let $2^i \leq n \leq 2^{i+1}$ and let T be a binary triangulation with n vertices. Observe that T splits into two binary triangulations T' and T'' with n_1 and n_2 vertices, $2^{i-1} \leq n_1, n_2 \leq 2^i$. By induction on i, T' and T'' can be transformed into sorted Δ triangulations in $O(n_1 \log n_1)$ and $O(n_2 \log n_2)$ flips. By Lemma 12 we can transform the resulting triangulation into a sorted $\Delta(1,2)$ triangulation with a linear number of flips. Our result follows. ∎

We proceed to prove the main result of this section:

Theorem 9 *Let T and T' be any labelled planar triangulation. Then T can be transformed into T' using $O(n \log n)$ flips, i.e. the diameter of G^l_T is at most $O(n \log n)$.*

Proof: To prove our result it is enough to show that T can be transformed to the sorted $\Delta(1,2)$ triangulation using $O(n \log n)$ flips. By Lemma 11, within $O(n)$ diagonal flips we can transform T into a not necessarily sorted $\Delta(1,2)$ triangulation T' whose three exterior vertices are v_1, v_2, v_n. Next using $O(n)$ flips, we can transform T' into a binary triangulation. To see that this is possible, take any *unlabelled* binary triangulation T'' and using a linear number of flips transform it into T' in such a way that v_1, v_2, and v_n remain in the exterior face of T'. By reversing the order in which these flips were performed, we can transform the labelled triangulation T' back into a labelled binary triangulation T'', thus producing a labelled binary triangulation. Finally by Lemma 12, we can transform T'' into the sorted $\Delta(1,2)$ triangulation. Our result follows. ∎

We will prove that any labelled triangulation with n vertices can be transformed into any other labelled triangulation using at most $O(n \log n)$ flips. We also prove that any planar triangulation with at least five vertices contains at least $n-2$ flippable edges. We show that this bound is tight. In the rest of this paper, all triangulations considered will be assumed to be labelled triangulations on $V = \{v_1, \ldots, v_n\}$.

5.2 The minimum vertex degree of G^l_T

Any planar triangulation contains a large number of flippable edges. In [17] it is proved that any triangulation of a set of n points in general position contains at least $\lceil \frac{n-4}{2} \rceil$ flippable edges. In this section we prove the corresponding result for planar triangulations, namely:

Theorem 10 *Any planar triangulation T with $n > 4$ vertices contains at least $n-2$ flippable edges. If T has minimum vertex degree at least 4, then T contains at least $\min\{2n+3, 3n-6\}$ flippable edges. Our bounds are tight.*

Proof: A triangle in a triangulation T is called *separating* if there are vertices inside as well as outside the triangle. Two edges are called *cofacial* if they belong to the boundary of a face of T. Let F ($\bar{F}$) be the set of flippable (nonflippable) edges in T. Define a relation $\mathcal{R} \subseteq \bar{F} \times F$ as follows:

$$(e, f) \in \mathcal{R} \iff e \in \bar{F}, f \in F, \text{ and } e \text{ and } f \text{ are cofacial.}$$

We claim that each nonflippable edge is related to *at least two* flippable edges. Let $e = v_i v_j$ be any nonflippable edge in T, and let $\{v_i, v_j, v_k\}$ and $\{v_i, v_j, v_l\}$ be the vertices of the two triangular faces of T incident with $v_i v_j$. Since $v_i v_j$ is nonflippable, v_k and v_l are adjacent in T. Since T has more than four vertices, vertices v_i and v_j cannot both have degree 3. If vertex v_i has degree at least 4, then both edges $v_i v_k$ and $v_i v_l$ are flippable; if vertex v_j has degree at least 4, then both edges $v_j v_k$ and $v_j v_l$ are flippable. On the other hand, each flippable edge is incident with exactly two faces, and hence is related to *at most four* nonflippable edges. Therefore we have:

$$2|\bar{F}| \leq |\mathcal{R}| \leq 4|F|.$$

Since the total number of edges in T is $3n-6$, it follows that the number of flippable edges is at least $(3n-6)/3 = n-2$.

Examples of planar triangulations that achieve this bound can be constructed as follows. Let T' be any planar triangulation with m vertices. Thus T' contains $2m-2$ triangular faces. Let T be the triangulation obtained as follows. In the middle of each of these triangular faces, insert a vertex adjacent to the vertices of the face. See Figure 19(a). It is easy to see that the only edges of T that are flippable are exactly the edges of T', i.e. $3m-6$ edges. On the other hand T contains exactly $m+2m-4 = 2m = 4$ vertices. Taking $n = 2m-4$ yields the desired result. This proves the first part of our theorem.

The argument in the previous paragraph shows that if T has more than four vertices and a nonflippable edge, then T contains a separating triangle.

Thus the second part of our result holds if T contains no separating triangles. Assume that T has minimum vertex degree at least 4 and T contains a separating triangle. The above argument shows that each nonflippable edge is related to exactly four flippable edges, i.e. $4|\bar{F}| = |\mathcal{R}|$. Also, if $\{v_i, v_j, v_k\}$ are the vertices of a face of T such that $v_i v_j$ is a flippable edge, then at least one of $v_i v_k$ and $v_j v_k$ is flippable. (Otherwise, the above argument implies that v_k has degree 3.) Hence each flippable edge is related to *at most two* nonflippable edges. Now we show that T contains at least 18 flippable edges which are not related to any nonflippable edge, and there are at least 3 extra flippable edges which are related to at most one nonflippable edge. Let $v_i v_j v_k$ be a separating triangle in T such that the triangulation $T(v_i v_j v_k)$, which consists of the triangle $v_i v_j v_k$ and all its interior vertices, contains no separating triangles. Since T has no vertex of degree 3, $T(v_i v_j v_k)$ contains at least 6 vertices. Since $T(v_i v_j v_k)$ contains no separating triangle, all edges of $T(v_i v_j v_k)$ are flippable in T. Therefore, all edges inside $v_i v_j v_k$ (there are least 9 such edges) are not related to any nonflippable edges. Similarly, T contains at least 9 flippable edges outside $v_i v_j v_k$ which are not related to any nonflippable edge. Notice also that each of the three edges $v_i v_j$, $v_j v_k$, $v_i v_k$ is related to at most one nonflippable edge. Thus we obtain:

$$4|\bar{F}| = |\mathcal{R}| \leq 2(|F| - 18 - 3) + 3.$$

Using $|\bar{F}| + |F| = 3n - 6$, we obtain $|F| \geq 2n + 2 + \frac{1}{2}$, i.e. $|F| \geq 2n + 3$.

Triangulations that achieve this bound can be obtained as follows. Let T' be a $\Delta(i, j)$ triangulation with $n - 6$ vertices. Insert a triangle in each of the two faces incident with v_i and v_j in such a way that the degree of the six new vertices is four; see Figure 19(b). The reader can easily verify that the resulting triangulation achieves the previous bound. Our result follows. ■

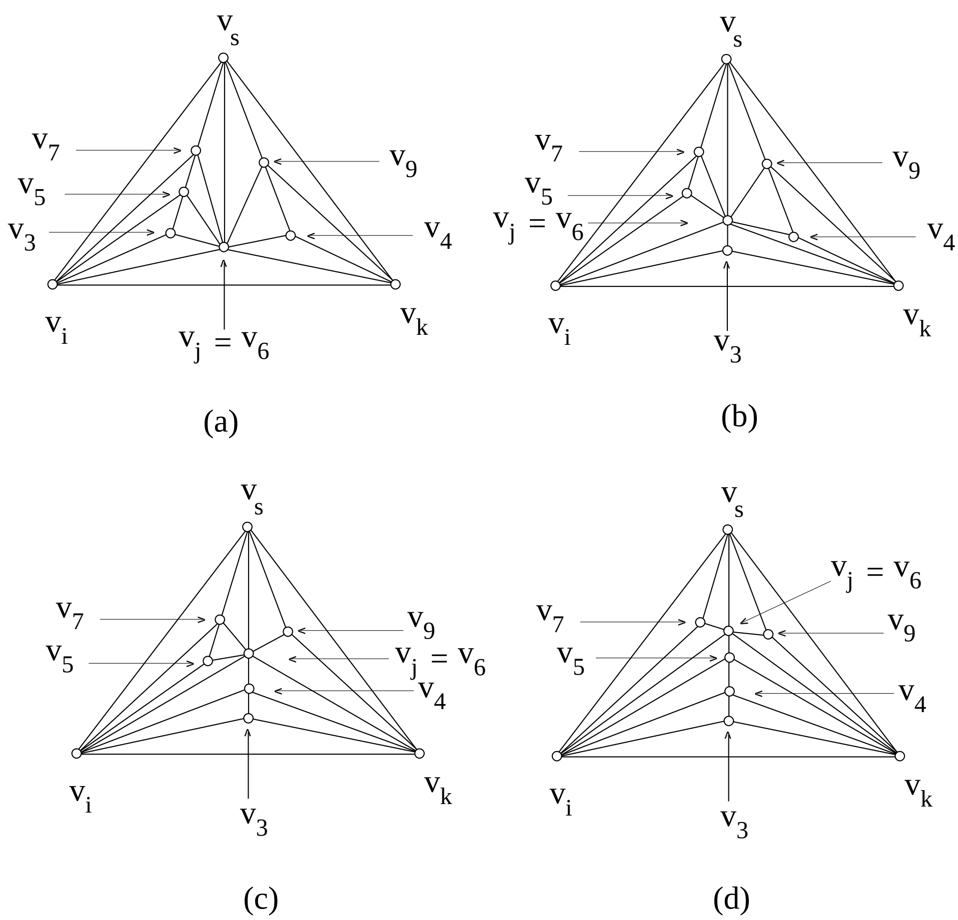

Figure 17: Illustrating Lemma 12.

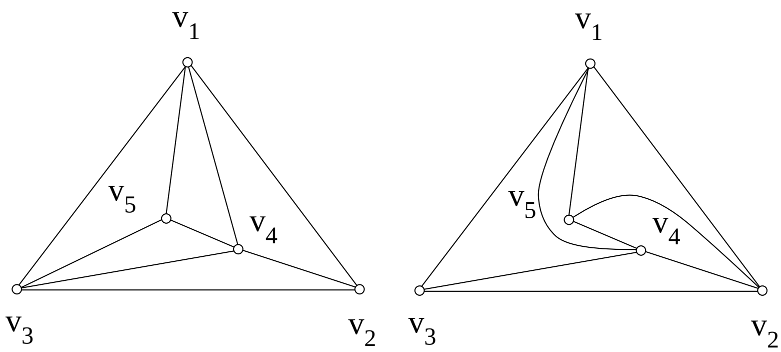

Figure 18: Two labelled triangulations at distance 2.

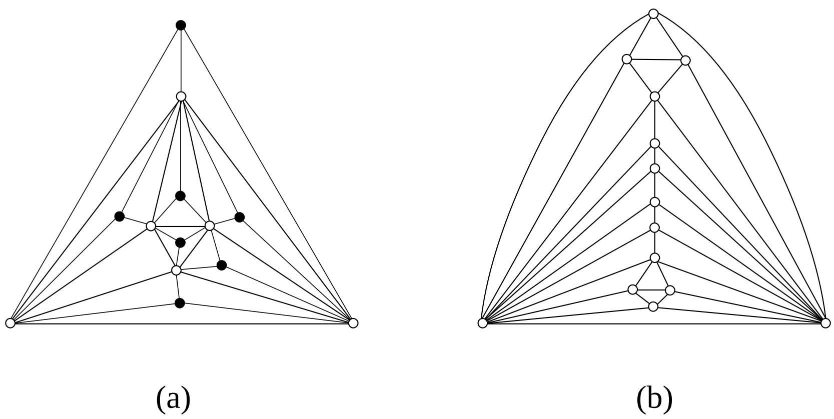

Figure 19: Two triangulations, the first with $n-2$ flippable edges, and the second with minimum degree 4 and $2n+3$ flippable edges.

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