Convex blocking and partial orders on the plane

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April 30, 2013

Abstract

Let $C = \{c_1, \ldots, c_n\}$ be a collection of disjoint closed bounded convex sets in the plane. Suppose that one of them, say $c_1$, represents a valuable object we want to uncover, and we are allowed to pick a direction $\alpha \in [0, 2\pi)$ along which we can translate (remove) the elements of $C$, one at a time, while avoiding collisions. We study the problem of finding a direction $\alpha_0$ such that the number of elements that have to be removed along $\alpha_0$ before we can remove $c_1$ is minimized. We prove that if we have the sorted set $D$ of directions defined by the tangents between pairs of elements of $C$, we can find $\alpha_0$ in $O(n^2)$ time. We also discuss the problem of sorting $D$ in $o(n^2 \log n)$ time.

1 Introduction

Consider a set $C = \{c_1, \ldots, c_n\}$ of pairwise disjoint closed bounded convex sets. It is well known that the elements of $C$ can be removed one at a time by translating them upwards while avoiding collisions with other elements of $C$; see [11, 16]. For example, the elements of the set $C = \{c_1, \ldots, c_9\}$ shown in Figure 1(a) can be removed in the order $c_2, c_3, c_1, c_9, c_4, c_5, c_7, c_8$. Clearly this result is also valid if we remove the elements of $C$ by translating them along any direction $\alpha \in [0, 2\pi)$.

Suppose that $c_1 \in C$ is a special object that we want to uncover, and that we are allowed to choose a direction $\alpha \in [0, 2\pi)$ along which we can remove the elements of $C$ one at a time while avoiding collisions. We want to find the

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The elements of $C$ can be removed in the upwards direction in the order $c_2, c_3, c_1, c_9, c_6, c_4, c_5, c_7, c_8$.

Two different directions in which we can remove elements from $C$.

Figure 1: Disassembly in different directions.

direction $\alpha_0$ that minimizes the number of elements we need to remove before we reach $c_1$. For example, in Figure 1(b), it is easy to see that if we remove the elements of $C$ in the direction $\alpha_1$, four elements of $C$ have to be removed before $c_1$ is uncovered, while for $\alpha_2$ we only need to remove two.

This problem can be seen as a variant of the problem known in computational geometry as the separability problem [5, 4, 9, 17]. Similar problems are studied in [1, 7], and it is also related to spherical orders determined by light obstructions [10].

In this paper, we present an $O(n^2)$ time algorithm to solve this problem, assuming that we have the sorted set $D$ of directions defined by the tangents between pairs of elements of $C$. To ease our presentation, in the remainder of the paper we will assume that the interior of the convex sets is not empty. It is not hard to see that the result holds for families of closed sets.

In Section 2 we give basic definitions and state the problem in these terms. In Section 3 we explain how we can reduce the search space of our problem to the set $D$ of critical directions. In Section 4 we present the data structure that we use to solve our problem. In Section 5, we present an algorithm to solve the main problem and we prove its time complexity. In Section 6, we discuss the difficulty of sorting $D$ in less than $O(n^2 \log n)$ time. Lastly, in Section 7 we present our conclusions.

2 Partial orders and blocking

Let $X$ be a finite set, and $<$ a relation on the elements of $X$ that satisfies the following conditions:

1. If $x < y$ and $y < z$ then $x < z$ (transitivity), and

2. $x \not< x$ (anti-reflexivity).
The set $X$ together with $<$ is called a partial order, and is usually denoted as $P(<, X)$.

Given $x, y \in X$, we say that $y$ covers $x$ if $x < y$ and there is no element $w \in X$ such that $x < w < y$. The diagram of $P(<, X)$ is the directed graph whose vertices are the elements of $X$, and which has an oriented edge from $x$ to $y$ if $y$ covers $x$.

We say that the diagram of $P(<, X)$ is planar if it can be drawn on the plane in such a way that the following conditions are satisfied:

a) the elements of $X$ are represented by points on the plane,

b) if $y$ is a cover of $x$, the edge joining them is a monotonically increasing curve (with respect to the $y$-axis) starting at $x$ and ending in $y$,

c) no edges of $P(<, X)$ intersect except perhaps at a common endpoint.

Given two elements $x, y \in X$, a supremum of $x, y$ is an element $w \in X$ such that $x < w$ and $y < w$, and for any other element $z \in X$ such that $x < z$ and $y < z$ we have that $w < z$. An infimum is defined in a similar way, except that we require $w$ to be $w < x$ and $w < y$. An ordered set is called a lattice if any two elements have a unique supremum and infimum. A lattice is called a planar lattice if its diagram is planar. Finally, a partial order $P(<, X)$ is called a truncated planar lattice if the order that results when both a least and a greatest element are added to $P(<, X)$ is a planar lattice.

Let $C = \{c_1, \ldots, c_n\}$ be a set of disjoint closed bounded convex sets on the plane and $\alpha \in [0, 2\pi)$. Given two convex sets $c_i$ and $c_j$ in $C$, we say that $c_j$ is an upper cover of $c_i$ in the direction $\alpha$ (for short, an $\alpha$-cover) if the following conditions are satisfied:

1. There is at least one directed line segment with direction $\alpha$ starting at a point in $c_i$ and ending at a point in $c_j$.

2. Any directed line segment with direction $\alpha$ starting at a point in $c_i$ and ending at a point in $c_j$ does not intersect any other element of $C$.

Clearly, if $c_j$ is an $\alpha$-cover of $c_i$, then to uncover $c_i$ along the $\alpha$ direction we need first to remove $c_j$. Observe that if $c_j$ is an $\alpha$-cover of $c_i$, then $c_i$ is an $(\alpha + \pi)$-cover of $c_j$. We say that $c_j$ blocks $c_i$ in the direction $\alpha$, written as $c_i \prec \alpha c_j$, if there is a sequence $c_i = c_{\sigma(1)}, c_{\sigma(2)}, \ldots, c_{\sigma(k)} = c_j$ of elements of $C$ such that $c_{\sigma(r+1)}$ is an $\alpha$-cover of $c_{\sigma(r)}$, with $r = 1, \ldots, k - 1$ (Figure 2). The following observation will be useful:

**Observation 2.1.** If $c_i \prec \alpha c_j$, then $c_j \prec \alpha + \pi c_i$.

Clearly if $c_i \prec \alpha c_j$ and $c_j \prec \alpha c_k$, then $c_i \prec \alpha c_k$. Since $c_j \not\prec \alpha c_i$, $C$ together with the blocking relation $\prec \alpha$ is a partial order $P(\prec \alpha, C)$. It is known that $P(\prec \alpha, C)$ is a truncated planar lattice [16].

Observe that the diagram of $P(\prec \alpha, C)$ has the elements of $C$ as vertices, and there is an oriented edge from $c_i$ to $c_j$ if $c_j$ is an $\alpha$-cover of $c_i$ (Figure 3). Since $\prec \alpha$ is defined using $\alpha$-coverings, the elements of $C$ that we need to remove
Figure 2: $c_{\sigma(r)}$ is an $\alpha$-cover of $c_{\sigma(r)}$, $r = 1, 2, 3$, and $c_{\sigma(r)} \prec_\alpha c_{\sigma(s)}$, $r < s$. In particular, $c_i \prec_\alpha c_j$.

in the $\alpha$ direction before an element $c_i$ of $C$ is reached are those convex sets $c_j$ such that $c_i \prec_\alpha c_j$. The set containing these elements will be called the upper set of $c_i$ in the $\alpha$ direction, or for short, the $\alpha$-up-set of $c_i$. Thus our problem reduces to that of finding the direction $\alpha_0$ such that the cardinality of the $\alpha_0$-up-set of $c_1$ is minimized.

Figure 3: Diagram of $P(\prec_\alpha, C)$ for $\alpha = \pi/2$.

Observe that as $\alpha$ changes, so does $P(\prec_\alpha, C)$. In fact, it is easy to find families of convex sets for which $P(\prec_\alpha, C)$ changes $\Theta(n^2)$ times.

In the next section we prove some properties of $P(\prec_\alpha, C)$ which will simplify the search space for $\alpha_0$.

3 The critical directions

A line $\ell$ is called a supporting line of a closed convex set $c$ if it intersects $c$, and $c$ is contained in one of the closed half-planes determined by $\ell$. In what follows, we will assume that no line is a supporting line of three or more elements of $C$, and that there are no two different parallel lines that each support two elements of $C$. 
Given two closed convex sets \( c_i \) and \( c_j \), a line \( \ell \) is called an **internal tangent** of \( c_i \) and \( c_j \) if \( \ell \) is a supporting line to both convex sets, and \( c_i \) is contained in one of the closed half-planes determined by \( \ell \) while \( c_j \) is contained in the other. Similarly, a line \( \ell \) is called an **external tangent** of \( c_i \) and \( c_j \) if \( \ell \) supports them and \( c_i \) and \( c_j \) are contained in the same closed half-plane determined by \( \ell \) (Figure 4).

Given \( c_i, c_j \in C \), if we orient their common supporting lines from \( c_i \) to \( c_j \), we can classify them as **left internal**, **right internal**, **left external**, and **right external** as in Figure 4. By definition, it is not hard to see that the internal tangents define critical directions where two convex sets can change their blocking relation.

![Diagram of internal and external tangents](image)

**Figure 4:** Internal and external tangents of \( c_i \) and \( c_j \).

Note that if \( \alpha \) is the direction defined by a tangent of \( c_i \) and \( c_j \), from \( c_i \) to \( c_j \), then \( \alpha + \pi \) is the direction of the same tangent of \( c_i \) and \( c_j \), but directed from \( c_j \) to \( c_i \).

**Observation 3.1.** There are at most 4\( \binom{n}{2} \) distinct values of \( \alpha \) where \( P(\preceq_\alpha, C) \) may change; these changes occur in the slopes defined by the internal tangents between pairs of elements of \( C \) (in both directions).

Given \( \alpha \in [0, 2\pi) \), and \( \beta = \alpha + \theta \), \( \theta \in [0, 2\pi) \), the interval \( I = [\alpha, \beta] \) will denote the set of directions \( \gamma \) such that \( \gamma = \alpha + \delta \), \( 0 \leq \delta \leq \theta \), addition taken mod 2\( \pi \). Note that we consider that said directions grow counter-clockwise.

Although the changes in \( P(\preceq_\alpha, C) \) may only happen at directions defined by internal tangents, we also consider directions defined by external tangents as they will be used in Section 4.

Let \( D = \{ \gamma_1, \ldots, \gamma_{8\binom{n}{2}} \} \) be the set of directions determined by the internal and external tangents of pairs of elements of \( C \), and suppose that they are labeled in such a way that for \( r < s \), \( \gamma_r < \gamma_s \). Observe that if we change the value of \( \alpha \) continuously from 0 to 2\( \pi \), \( P(\preceq_\alpha, C) \) may change only when \( \alpha \)
crosses an element of $\mathcal{D}$. Thus for any $\alpha$ and $\beta$ in the open interval $(\gamma_i, \gamma_{i+1})$, $P(\prec_{\alpha}, C) = P(\prec_{\beta}, C)$, $\gamma_i, \gamma_{i+1} \in \mathcal{D}$. Thus for any $\alpha \in (\gamma_i, \gamma_{i+1})$, $P(\prec_{\alpha}, C)$ will be denoted by $P(\prec_{\gamma_i}, C)$.

Lemma 3.3. If $c_i \prec_{\alpha} c_j$, then there is a direction $\beta \in (\alpha, \alpha + \pi)$ such that $c_i$ and $c_j$ are not comparable in $P(\prec_{\beta}, C)$.

Proof. By Observation 2.1, $c_j \prec_{\alpha + \pi} c_i$. Suppose that the lemma is false; then there is a critical direction $\gamma_i \in \mathcal{D}$, with $\alpha < \gamma_i < \alpha + \pi$, such that $c_i \prec_{\gamma_i} c_j$ and $c_j \prec_{\gamma_{i+1}} c_i$. Observe that at each critical direction in $\mathcal{D}$ in which the partial order changes, either two elements from $C$ stop being comparable or they become comparable. Thus if $c_i \prec_{\gamma_i} c_j$, then in $\gamma_{i+1}$ we have that $c_i \prec_{\gamma_{i+1}} c_j$, or $c_i$ and $c_j$ are incomparable. In either case it cannot happen that $c_j \prec_{\gamma_{i+1}} c_i$. \qed

We now prove:

Lemma 3.3. Let $c_i$ and $c_j$ be two convex sets in $C$. The set of directions in which $c_j$ blocks $c_i$ forms a unique non-empty interval $I_{i,j}$.

Proof. If $c_i \prec_{\alpha} c_j$ for some $\alpha$, by definition there is a sequence $S = c_{\sigma(1)}, c_{\sigma(2)}, \ldots, c_{\sigma(k)}$ of elements of $C$ such that $c_{\sigma(r+1)}$ is an $\alpha$-cover of $c_{\sigma(r)}$, with $c_i = c_{\sigma(1)}$, $c_j = c_{\sigma(k)}$ and $r = 1, \ldots, k-1$. Denote by $I_{\sigma(r), \sigma(r+1)}$ the interval of directions determined by the right and left interior tangents of $c_{\sigma(r)}$ and $c_{\sigma(r+1)}$, with $r = 1, \ldots, k-1$. Observe that $c_{\sigma(r+1)}$ is a $\gamma$-cover of $c_{\sigma(r)}$ for all $\gamma \in I_{\sigma(r), \sigma(r+1)}$.

The sequence $S$ determines then a set of directions $I(S)$ along which $c_i$ is blocked by $c_j$, where $I(S) = \bigcap_{r=1}^{k-1} I_{\sigma(r), \sigma(r+1)}$. Since at least $\alpha \in I(S)$ and the intersection of intervals is an interval, $I(S)$ is a non-empty interval.

When $c_j$ blocks $c_i$ in the direction $\alpha$, there could be more than one sequence determining such blocking. Any two such blocking sequences, $S$ and $S'$, differ at least in an element and, in general, $I(S_{i,j}) \neq I(S'_{i,j})$.

If we now consider all the directions $\delta \in [0, 2\pi)$ where $c_i \prec_{\delta} c_j$, then there is a finite number, $m \geq 1$, of distinct blocking sequences $S_1, \ldots, S_m$ given by those directions; and each $S_k$ determines a non-empty interval $I(S_k)$ of directions. Let $S = \{S_1, S_2, \ldots, S_m\}$. The set $I_{i,j}$ of directions in which $c_j$ blocks $c_i$, determined by all the sequences in $S$, is then the non-empty set $I_{i,j} = \bigcup_{k=1}^{m} I(S_k)$. If $I_{i,j}$ is in fact an interval, then our result holds.

By Lemma 3.2, there is a direction $\beta$ where $c_i \not\prec_{\beta} c_j$. Without loss of generality, suppose that $\beta = 0$; thus $0 \notin I(S_k)$ for each $S_k \in S$. Let $I(S_k) = [l_k, r_k]$ for each $S_k \in S$, and let $I = \{\theta_1, \theta_2\}$ where $\theta_1 = \min\{l_1, \ldots, l_m\}$ and $\theta_2 = \max\{r_1, \ldots, r_m\}$. We will show that $I_{i,j} = I$.

Clearly $I_{i,j} \subseteq I$, hence it remains to be proved that $I \subseteq I_{i,j}$. Let $\gamma \in [\theta_1, \theta_2] = I$, we will prove that $c_i \prec_{\gamma} c_j$, and therefore $I \subseteq I_{i,j}$. Let $\mathcal{B}_i$ be the band enclosed between the two supporting lines of $c_i$ in the $\gamma$ direction. We have three cases:

1. The convex set $c_j$ intersects $\mathcal{B}_i$. Thus, clearly $c_i \prec_{\gamma} c_j$. 
2. The convex set $c_j$ is to the left of $B_i$ (Figure 5).

Since $c_i \prec_{\theta_1} c_j$, we know that there is a sequence $c_i = c_{\sigma(1)}, c_{\sigma(2)}, \ldots$, $c_{\sigma(k_1)} = c_j$ such that $c_{\sigma(r+1)}$ is a $\theta_1$-cover of $c_{\sigma(r)}$ for $r = 1, \ldots, k_1 - 1$.

Since $c_{\sigma(2)}$ is a $\theta_1$-cover of $c_i = c_{\sigma(1)}$, there is a line segment parallel to the direction $\theta_1$ with endpoints in $c_i = c_{\sigma(1)}$ and $c_{\sigma(2)}$ such that it does not intersect any other convex set in $C$. Similarly, for $c_{\sigma(r)}$ and $c_{\sigma(r+1)}$ there is a line segment parallel to the direction $\theta_1$ with endpoints in $c_{\sigma(r)}$ and $c_{\sigma(r+1)}$, $2 \leq r \leq k_1 - 1$ such that it does not intersect any other convex set in $C$. Each $c_{\sigma(r)}$, $2 \leq r \leq k_1 - 1$ contains two endpoints from two of these line segments, and these endpoints can be joined with a line segment totally contained in $c_{\sigma(r)}$.

This forms a connected curve that starts in $c_i$ and ends in $c_j$, passing through all the elements of the sequence. This curve consists of two types of line segments: Those parallel to the $\theta_1$ direction, and those contained in $c_{\sigma(r)}$, $2 \leq r \leq k_1 - 1$. But $\theta_1 < \gamma$, so the first type always goes upwards and to the right of the $\gamma$ direction. The second type may go to the right or to the left; see (Figure 6).

By construction, such curve intersects $B_i$, and then at least one element of $\{c_{\sigma(2)}, \ldots, c_{\sigma(k_1-1)}\}$ also intersects $B_i$, say $c_{\sigma(s)}$, and thus $c_i \prec_{\gamma} c_{\sigma(s)}$.

Denote by $B_{\sigma(s)}$ the band bounded by the supporting lines of $c_{\sigma(s)}$ in the $\gamma$ direction. If $c_j$ intersects $B_{\sigma(s)}$ then $c_{\sigma(s)} \prec_{\gamma} c_j$, and by transitivity, $c_i \prec_{\gamma} c_j$.

Suppose then that $c_j$ does not intersect $B_{\sigma(s)}$. It is easy to see that $c_j$ should lie to the left of $B_{\sigma(s)}$. By substituting $c_{\sigma(s)}$ for $c_i$, and applying our previous argument repeatedly, we obtain a subsequence $\{c_i = c_{\sigma(i_1)}, c_{\sigma(i_2)}, \ldots, c_{\sigma(i_t)} = c_j\}$ of $\{c_{\sigma(1)}, c_{\sigma(2)}, \ldots, c_{\sigma(k_1)}\}$, with $i_1 < \cdots < i_t$, such that $c_{\sigma(i_1)} \prec_{\gamma} c_{\sigma(i_2)}, \ldots, c_{\sigma(i_{t-1})} \prec_{\gamma} c_{\sigma(i_t)}$, and thus $c_i \prec_{\gamma} c_j$.  

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**Figure 5:** $c_j$ to the left of $B_i$. 

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3. The convex set $c_j$ is to the right of the band $B_i$. The proof is analogous to that of the previous case, but now using the direction $\theta_2$ instead of $\theta_1$.

Since in all the three cases we have that $c_i \prec \gamma c_j$, then $I \subseteq I_{i,j}$, and therefore $I = I_{i,j}$, which is a unique non-empty interval where $c_j$ blocks $c_i$. □

In what follows, we will show how to maintain $P(\prec \alpha, C)$ as $\alpha$ changes from 0 to $2\pi$ in such a way that we can obtain $P(\prec \gamma_{k+1}, C)$ from $P(\prec \gamma_k, C)$, or more precisely, a triangulation $T_{k+1}$ from $T_k$ in constant time, $T_k$ to be defined in the following section. This will enable us to obtain $\alpha_0$ in $O(n^2)$.

4 $\alpha$-triangulations

We observe first that our problem can be solved by calculating the truncated lattices $P(\prec \gamma_k, C)$ for every direction $\gamma_k \in \mathcal{D}$, obtaining the $\gamma_k$-up-set of $c_i$ for each lattice, and then selecting a $\gamma_i \in \mathcal{D}$ which yields the smallest $\gamma_i$-up-set. Since calculating each $P(\prec \gamma_k, C)$ can be done in $O(n \log n)$ [16], and $\mathcal{D}$ has $8\binom{n}{2}$ elements, this yields an $O(n^3 \log n)$ time algorithm to solve our problem.

To improve this complexity, we will show that $P(\prec \gamma_{k+1}, C)$ can be obtained from $P(\prec \gamma_k, C)$ (more precisely, the $\gamma_k$-triangulation to be described shortly) in constant time.

For $\alpha \in [0, 2\pi)$, extend $P(\prec \alpha, C)$ to a planar lattice $P(\prec \alpha, C)$ by adding two special vertices, a source $s$ and a sink $t$ such that for each $c_i \in C$, $s \prec \alpha c_i \prec \alpha t$. We can picture $t$ (respectively $s$) as a very large convex set blocking all of the elements of $C$ (respectively blocked by all of the elements of $C$) in the direction $\alpha$ (Figure 7).

For each $\alpha$, we now extend $P'(\prec \alpha, C)$ to a triangulation $T_\alpha$; that is, a planar multigraph where every internal face—except the external one—is a triangle.
We will call $T_\alpha$ the $\alpha$-triangulation of $C$. To construct it, we use what we call $\alpha$-visibility.

Given two convex sets $c_i, c_j \in C$, we will say that $c_j$ is $\alpha$-visible from $c_i$ if there is an oriented line segment with direction $\alpha$ starting at a point on the boundary of $c_i$ and ending at a point on the boundary of $c_j$ such that it does not intersect any other convex set in $C$ (Figure 8). Such line segments will be called $(c_i, c_j)$ $\alpha$-visibility line segments.

If $c_j$ is $\alpha$-visible from $c_i$, the $\alpha$-visibility zone of $c_j$ and $c_i$ is the union of all the $(c_i, c_j)$ $\alpha$-visibility line segments (Figure 9). Note that the visibility zone of $c_i$ and $c_j$ is not necessarily a connected region; see Figure 10(a).

It is important to remark that if $c_j$ is $\alpha$-visible from $c_i$, then $c_i \prec_\alpha c_j$, however it is not necessarily true that $c_j$ is an $\alpha$-cover of $c_i$. On the other hand, if $c_j$ is an $\alpha$-cover of $c_i$, then $c_j$ is $\alpha$-visible from $c_i$ and their $\alpha$-visibility zone is not empty and connected.

To obtain $T_\alpha$ we proceed as follows: If $c_j$ is $\alpha$-visible from $c_i$, and $c_j$ is not an $\alpha$-cover of $c_i$, then we add to $P'(\preceq_\alpha, C)$ an oriented arc from $c_i$ to $c_j$ for each connected component of the $\alpha$-visibility zone of $c_i$ and $c_j$. Each of these
Figure 9: The shaded region is the $\alpha$-visibility zone of $c_j$ from $c_i$.  

arcs can be drawn passing through their corresponding connected component of the $\alpha$-visibility zone of $c_j$ and $c_i$. Clearly $\mathcal{T}_\alpha$ is planar, and the above procedure yields an embedding of $\mathcal{T}_\alpha$ on the plane such that all of its faces, except for the external face, are triangular faces; see Figures 10(a), 10(b).

The extra arcs added to $P'(\llangle \alpha, C \rrangle)$, will be called $\alpha$-visibility arcs, to distinguish them from the regular arcs of $P'(\llangle \alpha, C \rrangle)$. In particular, for each direction $\alpha$, the source $s$ and the sink $t$ will always be joined by two visibility arcs bounding the external face of $\mathcal{T}_\alpha$. In all our figures, $\alpha$-visibility arcs will be drawn with dashed curves, and the arcs from $P'(\llangle \alpha, C \rrangle)$ with solid line segments. The triangulation $\mathcal{T}_\alpha$ arising from the lattice shown in Figure 7 is given in Figure 11.

Figure 10: Visibility zone and its corresponding oriented multi-edge.

Observe that all the arcs in $\mathcal{T}_\alpha$ belong to two triangular faces of $\mathcal{T}_\alpha$, except for two arcs connecting $s$ and $t$. Let $e$ be an arc of $\mathcal{T}_\alpha$ that belongs to two triangular faces $f$ and $f'$ of $\mathcal{T}_\alpha$. Each of these faces contains a vertex (an element of $C$) that is not an endpoint of $e$. These elements will be called opposite elements with respect to $e$.

Since $\mathcal{D}$ has $8\binom{n}{2}$ elements, there are at most $8\binom{n}{2}$ triangulations $\mathcal{T}_\alpha$, $\alpha \in \mathcal{D}$. We now study how to obtain $\mathcal{T}_{\gamma_{k+1}}$ from $\mathcal{T}_{\gamma_k}$. We remark first that there are
many cases in which $T_\gamma$ and $T_{\gamma+1}$ are the same; see Figure 12. It is also possible that $T_\gamma \neq T_{\gamma+1}$, but $P'(\prec_\gamma, C) = P'(\prec_{\gamma+1}, C)$. This could happen if $T_{\gamma+1}$ differs from $T_\gamma$ only in visibility arcs.

Let $c_i$ and $c_j$ be the elements of $C$ that define $\gamma_{k+1}$. By definition, $\gamma_{k+1}$ is parallel to one of the four tangents defined by $c_i$ and $c_j$. If $T_\gamma$ differs from $T_{\gamma+1}$, then either $\gamma_{k+1}$ is defined by an external tangent, and this caused a visibility change, or $\gamma_{k+1}$ is defined by an internal tangent, and this caused a change in the partial order.

We will now prove that the difference between the triangulations $T_\gamma$ and $T_{\gamma+1}$ (if any) will be an arc flip, as defined in [14]; that is, we will remove an arc $e$ from $T_\gamma$, and replace it by another arc connecting two elements of $C$ which are opposite with respect to $e$.

**Lemma 4.1.** Let $\gamma_k, \gamma_{k+1} \in D$, and let $c_i$ and $c_j$ be the convex sets defining $\gamma_{k+1}$. If $T_\gamma \neq T_{\gamma+1}$, then $T_{\gamma+1}$ can be obtained from $T_\gamma$ by flipping an arc in $T_\gamma$. Such an arc flip involves arcs incident to $c_i$, $c_j$, or both of them.
**Proof.** Without loss of generality suppose that $\gamma_{k+1} = \frac{\pi}{2}$ and that the tangent defining $\gamma_{k+1}$ is oriented from $c_i$ to $c_j$. Such tangent can be left external, right external, left internal, or right internal. However, the analysis for the case when $\gamma_{k+1}$ is defined by a right external tangent is equivalent to the one for the left external case, and the same applies to the right internal and left internal cases.

- $\gamma_{k+1}$ is defined by a left external tangent $\ell$. Since $T_{\gamma_k} \neq T_{\gamma_{k+1}}$, no other element of $C$ intersects $\ell$ between $c_i$ and $c_j$ (Figure 13(a)). Therefore there is a $\gamma_{k+1}$-visibility arc from $c_i$ to $c_j$, and there is also a $\gamma_k$-visibility arc from $c_i$ to $c_j$.

Let $c_a$ be the first element below $c_i$ that intersects $\ell$, and $c_b$ the first element above $c_j$ that intersects $\ell$. If no element of $C$ intersects $\ell$ below $c_i$, then $c_a = s$; similarly, if no element of $C$ intersects $\ell$ above $c_j$, then $c_b = t$.

It is not hard to see that $c_j$ is $\gamma_k$-visible from $c_a$ but not $\gamma_{k+1}$-visible from it (to the left of $c_j$) because $c_i$ blocks any line segment parallel to $\gamma_{k+1}$ between them. Also, $c_b$ is $\gamma_{k+1}$-visible from $c_i$ and not $\gamma_k$-visible from $c_i$ (to the left of $c_j$) because $c_j$ gets in the way of visibility. Finally, $c_b$ is $\gamma_k$- and $\gamma_{k+1}$-visible from $c_j$, $c_i$ is $\gamma_k$- and $\gamma_{k+1}$-visible from $c_a$, and $c_a$ and $c_b$ are $\gamma_k$- and $\gamma_{k+1}$-visible (Figure 13(b)).

\[\ell\]
\[\begin{array}{c}
\circ \quad c_j \\
\circ \quad c_i \\
\circ \quad c_b \\
\circ \quad c_a
\end{array}\]

(a) There is no change in visibility for the direction $\gamma_{k+1}$ when there is an element between $c_i$ and $c_j$.

\[\begin{array}{c}
\circ \quad c_b \\
\circ \quad c_j \\
\circ \quad c_i \\
\circ \quad c_a
\end{array}\]

(b) The quadrangle defined by $c_i$, $c_j$, $c_a$, and $c_b$ and the flip when going from $\gamma_k$ to $\gamma_{k+1}$.

**Figure 13:** The case when $\gamma_{k+1}$ is a left external tangent.

In other words, the elements $c_i$, $c_j$, $c_a$, and $c_b$ form a quadrangle in both $T_{\gamma_k}$ and $T_{\gamma_{k+1}}$, with the $\gamma_k$-visibility arc from $c_a$ to $c_j$ being a diagonal in $\gamma_k$ of such a quadrangle, and this diagonal flips to the $\gamma_{k+1}$-visibility arc from $c_i$ to $c_b$ in $T_{\gamma_{k+1}}$.

- $\gamma_{k+1}$ is defined by a left internal tangent $\ell$. Since $T_{\gamma_k} \neq T_{\gamma_{k+1}}$, no other element of $C$ intersects $\ell$ between $c_i$ and $c_j$ (Figure 14(a)). Therefore there is a regular arc, defined in $P'(<\gamma_k, C)$, from $c_i$ to $c_j$. 

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Let \( c_a \) be the first element below \( c_i \) that intersects \( \ell \), and \( c_b \) the first element above \( c_j \) that intersects \( \ell \). If no element of \( C \) intersects \( \ell \) below \( c_i \), then \( c_a = s \); similarly, if no element of \( C \) intersects \( \ell \) above \( c_j \), then \( c_b = t \).

It is not hard to see that \( c_j \) is a \( \gamma_k \)-cover of \( c_i \), but \( c_j \) is not a \( \gamma_{k+1} \)-cover of \( c_i \), and the arc from \( c_i \) to \( c_j \) in \( T_{\gamma_k} \) is replaced by the \( \gamma_{k+1} \)-visibility arc from \( c_a \) to \( c_b \) in \( T_{\gamma_{k+1}} \). Finally, \( c_b \) is \( \gamma_k \)- and \( \gamma_{k+1} \)-visible from \( c_i \) and \( c_j \), and \( c_i \) and \( c_j \) are \( \gamma_k \)- and \( \gamma_{k+1} \)-visible from \( c_a \) (Figure 14(b)).

**Figure 14:** The case when \( \gamma_{k+1} \) is a left internal tangent.

In other words, the elements \( c_i, c_j, c_a, \) and \( c_b \) form a quadrangle in both \( T_{\gamma_k} \) and \( T_{\gamma_{k+1}} \), with the arc from \( c_i \) to \( c_j \) in \( T_{\gamma_k} \) being a diagonal of such quadrangle, and this diagonal flips to the \( \gamma_{k+1} \)-visibility arc from \( c_a \) to \( c_b \) in \( T_{\gamma_{k+1}} \).

Therefore \( T_{\gamma_{k+1}} \) can be obtained from \( T_{\gamma_k} \) by performing an arc flip, and our result holds. Even more, in each case we know if the arc to flip is an \( \alpha \)-visibility arc or a regular arc, and if it flips to a regular arc or to an \( \alpha \)-visibility arc.

Figure 15 shows an example of the arc flip performed to transform \( T_{\gamma_k} \) into \( T_{\gamma_{k+1}} \).

We will assume that for each direction \( \gamma \) in \( D \), we also have associated to it the two convex sets in \( C \) that define it. The next result follows:

**Corollary 4.2.** Given \( T_{\gamma_k} \), we can obtain \( T_{\gamma_{k+1}} \) in \( O(1) \) time.

## 5 An algorithm to find \( \alpha_0 \)

In this section we prove that if we have the elements of \( D \) sorted then we can find the direction \( \alpha_0 \) for which the up-set of \( c_1 \) is minimized in \( O(n^2) \) time.
We will need two lemmas to prove our result.

**Lemma 5.1.** For any element $c_i \in C$, as we go from $\gamma_1$ to $\gamma_{8(\frac{n}{2})}$, the up-set of $c_i$ changes $O(n)$ times.

**Proof.** Let $c_i, c_j \in C$, $c_i \neq c_j$. By Lemma 3.3, the set of directions for which $c_j$ blocks $c_i$ is an interval $I_{i,j}$. This means that as we go from $\gamma_1$ to $\gamma_{8(\frac{n}{2})}$, $c_j$ enters and leaves the up-set of $c_i$ once. Therefore the up-set of $c_i$ changes a linear number of times.

Suppose next that for a direction $\gamma_k \in D$ we have $T_{\gamma_k}$ such that it vertices are colored as above, that $c_1$ and all the elements of $C$ that belong to the up-set of $c_1$ are colored red, and the remaining elements of $C$ are colored blue. We now show how we can detect in constant time whether the up-set of $c_1$ changes.

**Lemma 5.2.** Given $T_{\gamma_k}$ such that it vertices are colored as above, we can detect whether the up-set of $c_1$ changes in $T_{\gamma_{k+1}}$ in constant time.

**Proof.** Observe that if $\gamma_{k+1}$ is defined by an external tangent of two elements $c_i, c_j \in C$, then $P'(\prec_{\gamma_k}, C) = P'(\prec_{\gamma_{k+1}}, C)$, and therefore the up-set of $c_1$ remains unchanged, and the coloring of $C$ for $\gamma_{k+1}$ is the same as that for $\gamma_k$.

Suppose then that $\gamma_{k+1}$ is defined by an internal tangent of two elements $c_i$ and $c_j$ of $C$. Two cases arise depending on whether $\gamma_{k+1}$ is a left or a right tangent of $c_i$ and $c_j$.

Suppose first that $\gamma_{k+1}$ is a right internal tangent. In this case, $c_i$ and $c_j$ are comparable in $T_{\gamma_{k+1}}$. Assume without loss of generality that $c_i \prec_{\gamma_{k+1}} c_j$. If $c_i$ and $c_j$ were comparable in $T_{\gamma_k}$, then $P'(\prec_{\gamma_k}, C) = P'(\prec_{\gamma_{k+1}}, C)$ and the up-set of $c_1$ does not change.

Suppose then that $c_i$ and $c_j$ are not comparable in $T_{\gamma_k}$. Observe first that if $c_i$ is red, and $c_j$ is blue, then $c_j$ becomes comparable to $c_1$, and the up-set of $c_1$
changes. In all the other cases when $\gamma_{k+1}$ is a right internal tangent, the up-set of $c_1$ remains unchanged.

Suppose next that $\gamma_{k+1}$ is a left internal tangent of $c_i$ and $c_j$. In this case, it must happen that $c_i$ and $c_j$ are comparable in $T_{\gamma_k}$. Assume that $c_i \prec_{\gamma_k} c_j$. If both $c_i$ and $c_j$ are red, then $c_j$ could leave the up-set of $c_1$, but only if it was a $\gamma_k$-cover of $c_i$. The case when $c_i$ is red and $c_j$ is blue in $T_{\gamma_k}$ cannot happen, since $c_i \prec_{\gamma_k} c_j$. In all the other cases when $\gamma_{k+1}$ is a left internal tangent, the up-set of $c_1$ remains unchanged.

Therefore, the up-set of $c_1$ can change only when $\gamma_{k+1}$ is a right internal tangent and $c_i$ is red and $c_j$ is blue; or when $\gamma_{k+1}$ is a left internal tangent and both $c_i$ and $c_j$ are red, and $c_j$ is a $\gamma_k$-cover of $c_i$. We can test either case in constant time. For the second case, we can check if $c_j$ is a $\gamma_k$-cover of $c_i$, and by Lemma 4.1 we know before each flip if that is the case.

Observe that if the up-set of $c_1$ does not change, then the red and blue coloring of the elements of $T_{\gamma_k}$ is maintained in $T_{\gamma_{k+1}}$ in the sense that the red elements in $T_{\gamma_k}$ are the elements in the up-set of $c_1$ in $T_{\gamma_{k+1}}$.

**Theorem 5.3.** Suppose that we have the sorted set of directions $D = \{\gamma_1, \ldots, \gamma_8(\frac{n}{2})\}$, and that for each $\gamma_k$, we are also given the pair of elements $c_i$ and $c_j$ that generated it. Then we can find $\alpha_0$ in $O(n^2)$.

**Proof.** Construct $P'(<\gamma_1, C)$ and $T_{\gamma_1}$ in $O(n \log n)$ time. Next we calculate the $\gamma_1$-up-set of $c_1$ in $O(n)$ time by using BFS on $P'(<\gamma_1, C)$.

By Corollary 4.2, we can obtain, one by one, the $T_{\gamma_1}, \ldots, T_{\gamma_8(\frac{n}{2})}$ in overall quadratic time. By Lemma 5.2, we can find, also in overall quadratic time, the set of directions in which the up-set of $c_1$ changes. By Lemma 5.1, the up-set of $c_1$ changes a linear number of times. Each time this happens, we recolor the elements of our current partial order in linear time. Thus we can also maintain the coloring of the vertices of $T_{\gamma_1}, \ldots, T_{\gamma_8(\frac{n}{2})}$ in quadratic time.

Therefore we can find $\alpha_0$ in $O(n^2)$ time. \qed

### 6 Some remarks about sorting $D$

If we assume that for each pair of elements of $C$, we can calculate their tangent lines in constant time, then we can sort the elements of $D$ in $O(n^2 \log n)$ time.

A similar problem to that of sorting the elements of $D$ arises from the problem of sorting the intersections generated by arrangements of curves on the plane.

A family of $x$-monotone Jordan curves is called well behaved if each time two curves intersect they cross each other, and any two curves intersect at most $s$ times, where $s$ is constant ([12], pages 399 and 404). In this context, it is also assumed that the intersections of any two curves can be calculated in constant time; this is usually referred to as being under a proper model of computation ([3, 15]).
It is known that for arrangements of well behaved curves with \( n \) elements in which any two of them intersect at most two times, the arrangement generated by them, including the set of all of their intersections, can be constructed in \( O(n^2 \cdot 2^{o(n)}) \), where \( o(n) \) is the inverse of the Ackermann function \([8]\). However it is not known how to sort these intersections according to the \( x \)-axis in \( o(n^2 \log n) \) time, even when we consider arrangements of lines.

The well known sorting \( X + Y \) open problem (Problem 41 in \([6]\)) says: Given two sets \( X \) and \( Y \) of numbers, each of size \( n \), how quickly can the set \( X + Y \) of all pairwise sums be sorted? In \([13]\) it is proved that the sorting \( X + Y \) problem is a particular case of the problem of sorting the intersections of an arrangement of lines according to the \( x \)-axis. The first reference to the sorting \( X + Y \) problem was made in 1976, and it remains open: By the result proved in \([13]\), sorting the intersections of an arrangement of lines according to the \( x \)-axis is an even stronger result.

In what follows, and to ease our presentation, we assume that the boundary of the elements of \( C \) is smooth. This avoids an unenlightening case analysis that leaves our results unchanged. To see that our problem can be reduced to that of sorting the intersection points of arrangements of well behaved curves in which any two of them intersect at most twice, we proceed as follows.

Let \( c_i \in C \), and let \( U_i \) and \( L_i \) be the upper and lower chains of the boundary of \( C \). Under the dual transformation which maps a non-vertical line \( \ell \) defined by the equation \( y = mx - n \) to the dual point \( \ell^* = (m, n) \) and a point \( p = (a, b) \) to the line \( p^* : y = ax - b \), the points in \( U_i \) will be mapped to lines whose lower envelope will be a concave \( x \)-monotone curve that we will call \( U_i^* \), and the points in \( L_i \) will be mapped to lines whose upper envelope will be a convex \( x \)-monotone curve that we will call \( L_i^* \).

In the dual space, every line that intersects \( c_i \) is mapped to a point bounded from above by \( L_i^* \) and from below by a \( U_i^* \), and every point inside \( c_i \) is mapped to a line enclosed between \( U_i^* \) and \( L_i^* \) respectively (\([2, Section 7.4]\), (Figure 16).

![Figure 16: An element \( c_i \) of \( C \) and its mapping in the dual space.](image)

Let \( c_i, c_j \in C \). If a line \( \ell \) is a tangent of \( c_i \) and \( c_j \), then it intersects both convex sets. Without loss of generality, suppose that it does so in \( U_i \) and \( L_j \). In the dual space this results in \( \ell^* \) being the intersection point of \( U_i^* \) and \( L_j^* \). For simplicity, we will assume that there are no vertical tangents between pairs of elements in \( C \); we can always slightly rotate the whole set if necessary.

Let \( \Gamma = \{U_i^*, L_i^*|i = 1, \ldots, n\} \) be an arrangement of curves. The next lemma
follows:

**Lemma 6.1.** Any two curves in $\Gamma$ intersect at most twice.

*Proof.* Let $c_i \in C$, and let $\ell$ be a non-vertical line tangent to $c_i$. Observe that if $\ell$ intersects $c_i$ at a point in $U_i$, then $c_i$ lies below $\ell$; if $\ell$ intersects the boundary of $c_i$ at a point in $L_i$, $c_i$ is above $\ell$.

Let $c_i, c_j \in C$, and let $\ell$ be a line tangent to both of them. If $\ell$ is an external tangent, then $c_i$ and $c_j$ are both contained in the same closed half-plane determined by $\ell$. Therefore $\ell$ intersects $c_i$ in $L_i$ (respectively $U_i$), and $\ell$ intersects $c_j$ in $L_j$ (respectively $U_j$) (Figure 17).

If $\ell$ is an internal tangent to $c_i$ and $c_j$, then one of them lies above $\ell$ and the other below it. Thus if $\ell$ intersects $c_i$ in $U_i$ (resp. $L_i$), it intersects $c_j$ in $L_i$ (resp. $U_i$).

![Figure 17: Intersections of a tangent $\ell$ with the upper and lower chains of two convex sets in $C$.](image)

We now prove that any two $\tau^*_i, \tau^*_j \in \Gamma$ intersect at most twice. Suppose on the contrary that they intersect at least three times. Assume that $\tau^*_i, \tau^*_j$ were generated by upper or lower chains of two elements $c_i', c_j'$ of $C$. Since the intersection points of $\tau^*_i, \tau^*_j$ correspond to common tangents of $c_i', c_j'$, one of these tangents is an internal tangent, and the other an external tangent of $c_i', c_j'$. But an internal tangent touches a lower and an upper chain of $c_i', c_j'$, and an external chain intersects either two lower or two upper chains of $c_i', c_j'$. Thus $\tau^*_i, \tau^*_j$ intersect at most twice. 

There are two ways in which the curves $U^*_i, L^*_i, U^*_j, L^*_j$ can intersect: If $c_i$ is not contained in the vertical strip defined by the vertical tangents to $c_j$ (or vice versa as in Figure 18(a)), then each pair of curves will intersect at most once (Figure 18(b)). If $c_i$ is contained in the vertical strip defined by the vertical tangents to $c_j$ (or vice versa; Figure 18(c)), then each pair of curves will intersect at most twice (Figure 18(d)).

Therefore the problem of calculating and sorting $D$ is equivalent to calculating and sorting the intersections of an arrangement of curves that intersect each other at most twice.
Figure 18: Top: Each pair of curves intersects at most once in the dual space. Bottom: Each pair of curves intersects at most twice in the dual space.

7 Conclusions

In this paper we studied a variant of the classic separability problem. Given a set \( C = \{c_1, \ldots, c_n\} \) of pairwise disjoint closed convex sets in the plane, find a direction \( \alpha_0 \) minimizing the number of elements of \( C \) that have to be removed, along the direction \( \alpha_0 \), in order to reach a particular element \( c_1 \in C \). We present an \( O(n^2) \) time algorithm to solve this problem, under the assumption we have the sorted set \( D \) of slopes of tangents to pairs of elements of \( C \).
References


