A problem on hinged dissections with colours

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Abstract

We examine the following problem. Given a square C we want a hinged dissection of C into congruent squares and a colouring of the edges of these smaller squares with k colours such that we can transform the original square into another with its perimeter coloured with colour i, for all i in $\{1, \cdots, k\}$. We have the restriction that the moves have to be realizable in the plane, so when swinging the pieces no overlapping are allowed. We show that for k colours, we need p^2 pieces, with p an even number and at least $2k + 2\sqrt{k^2 - k}$, this by using a necklace made of the p^2 pieces and a ingenious way to wrap it into a square.

1 Introduction

A geometric dissection is a cutting of a geometric figure into pieces that we can rearrange to form another figure. The hinged dissection problem ask if given two geometric figures A and B of the same area, A can be dissected and fixed with hinges at some joints so that: first A is still in one piece and second A can be folded into B by swinging the pieces around the hinges. There is a well known and beautiful solution when A is a equilateral triangle and B is a square due to Henry E. Dudeney, see [10, 11]. This problem has a long history and we referred the reader to [1, 2, 11, 12] to see a sample of the diversity of results in this topic. Here we consider a related problem. Given

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a square C we want a hinged dissection of C, where the pieces are squares, and a colouring of the edges of the pieces with k colours such that we can unfold the original square and fold it back but with its perimeter coloured with colour i, for all i in $\{1, \cdots, k\}$. We observe that the moves have to be realizable in the plane, so when swinging the pieces no overlapping are allowed. We call this the hinged dissection problem with k colours, denoted HDP_k.

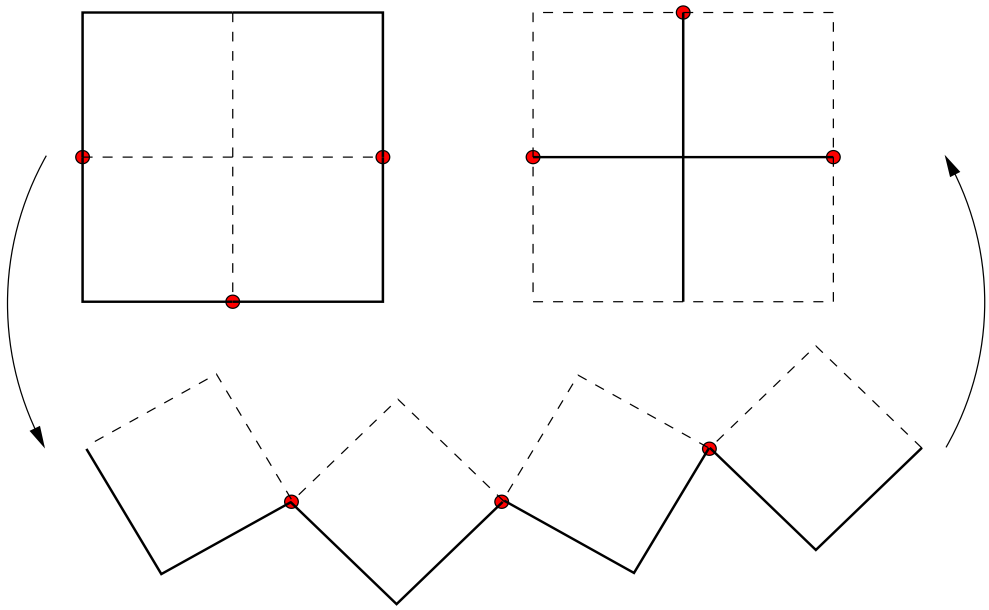

Figure 1: A solution for the hinged dissection problem with 2 colours.

The case when $k = 2$ has an easy solution and is best seen in Figure 1, where colour 1 is represented by a solid line and colour 2 by a dashed line. For the general case we have two solutions. Section 2 and Section 3 contain a description of two structures, one called necklace and the other chain, which are the basic structures for our solutions. We also include some of their combinatorial properties. In Subsection 4.1 we present a solution that has a nice combinatorial flavour associated to it. The second solution is outline in Subsection 4.2. Section 5 provided some lemmas dealing with the swinging of pieces. Section 6 comments on a linear time algorithm to check if a given hinged structure can be folded into a line. Finally, the last section contain some combinatorial geometry problems.

The square lattice L_n is a well known graph [4, 15] but for reference we include here a definition. The graph L_n has as vertices the set $\{0, \ldots, n-1\} \times \{0, \ldots, n-1\}$ and where two vertices (i, j) and (i', j') are adjacent if $|i - i'| + |j - j'| = 1$. For the basic graph theory required in this paper the reader is referred to [5, 19].

2 Necklaces and chains

Here we describe the basic idea behind our first solution. Given n congruent squares $S_1, \cdots S_n$ we join S_i to S_{i-1} and S_{i+1} by two hinges fixed at two diagonally opposite points of S_i. If S_1 is also hinged to S_n, we called this structure a *necklace* of size n, $\mathcal{NK}_n$. Otherwise, we called the structure a *chain* of size n, $\mathcal{CH}_n$. See Figure 2 for examples. We assume that each S_i comes with a marked diagonal which shows where are the hinges in S_i.

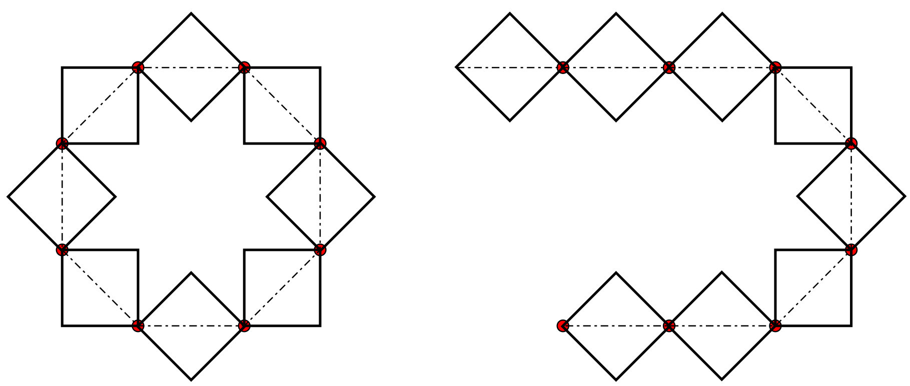

Figure 2: The necklace $\mathcal{NK}_8$ and the chain $\mathcal{CH}_8$.

Our goal is to fold a necklace $\mathcal{NK}_n$ into a square C. The first obvious requirement is $n = p^2$ for some p, and then, it follows that the perimeter of C will be formed with $4(p-1)$ pieces from $\mathcal{NK}_{p^2}$. Let $k > 3$ be an integer and suppose that p is big enough so that we can have at least k segments of $4(p-1)$ pieces in $\mathcal{NK}_{p^2}$. For that we require p to be at least $2k + 2\sqrt{k^2 - k}$.

We describe now a plausible solution to the hinged dissection problem with k colours. We colour each of the k segments with a different colour, suppose that we can fold $\mathcal{NK}_{p^2}$ into C in such a way that a monochromatic segment is on the perimeter, the election of the monochromatic segment is irrelevant due to the symmetry of $\mathcal{NK}_{p^2}$. Thus, we get a solution for HDP_k.

Therefore, we only require a way to fold a necklace $\mathcal{NK}_{p^2}$ into a square with the desired condition that a segment in $\mathcal{NK}_{p^2}$ is on the perimeter.

3 Folded necklaces

In this section we study ways to fold a necklace. A *basic move* is either a swing of two squares around a hinge or a translation of a square in the plane. A *move* is a group of basic moves perform simultaneously. See Section 5 for examples. A *way to fold* a necklace $\mathcal{NK}_{p^2}$ into a square C is a sequence α of moves. The result is a distribution of the p^2 pieces of $\mathcal{NK}_{p^2}$ on C and in each piece there are marks for the places of its two hinges. Thus, from the labels on the pieces and the marks of the hinges, we know which adjacent pieces are joint together. We call this distribution of the S_i with their marks a *folded* $\mathcal{NK}_{p^2}$, or just folded necklace, and it is denoted by $F_\alpha(\mathcal{NK}_{p^2})$.

For us 2 folded necklaces are the same if they just differed by a relabelling of the squares S_i induced by a cyclic permutation of $(1,\ldots,p^2)$, then, 2 different sequences α_1 and α_2 of moves may lead to the same folded $\mathcal{NK}_{p^2}$.

Each folded necklace $F = F_\alpha(\mathcal{NK}_{p^2})$ has two associated graphs. First, we call the places where there are corners of squares in the partition of F *crossing points.* Observe that there are $(p+1)^2$ crossing points in F. Now, we have the graph with vertex set the crossing points of F and there is an edge between two vertices if they are consecutive crossing points either vertical or horizontally in the dissection. This graph is clearly the square lattice L_{p+1}. With this planar representation, there is a natural bijection between the internal faces of L_{p+1} and the squares S_i of $\mathcal{NK}_{p^2}$.

The second graph is described by the set of segments joining two hinges. That is, there is a vertex for each crossing point in the folding necklace where there are one or two hinges, and there is an edge between crossing points a and b if there are hinges h_1 in a and h_2 in b such that h_1 and h_2 are in the same square S_i, for some i. Called this graph H_p.

We have some basic facts about H_p. By definition of the graphs, $V(H_p) \subseteq V(L_{p+1})$. The degree of any vertex in H_p is either 2 or 4, so H_p is Eulerian. This also follows as H_p comes from a folded necklace which clearly induces a Eulerian decomposition of H_p. Any two adjacent vertices v and u in H_p are at distance two in L_{p+1}, as the edge $\{u, v\}$ is a diagonal in some internal face of L_{p+1}. Finally, the four corners of a folding necklace $F(\mathcal{NK}_{p^2})$ cannot be places for hinges.

Proposition 3.1. *The graph H_p has as vertex set,*

$$V(H_p) = \{(i,j) \,|\, 0 \le i, j \le p,\, i+j \text{ odd}\}$$

Corollary 3.2. *A necessary condition for a necklace $\mathcal{NK}_{p^2}$ to have an associated folding necklace is p to be an even number.*

Thus, we can identify H_p as the graph with vertices $\{(i,j)|0 \le i,j \le p, \quad i+j \quad \text{is odd}\}$ and two vertices are adjacent if they are at distance 2 in L_{p+1}. See Figure 3.

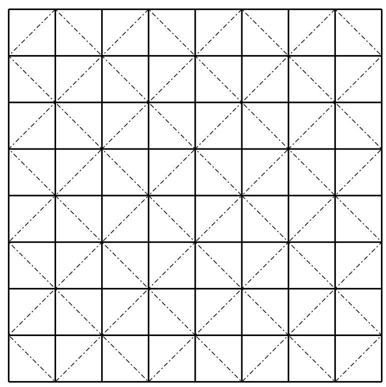

Figure 3: The graphs L_9, in solid line. and H_8, in dotted line.

As we mentioned, H_p is Eulerian, so it admits an Eulerian decomposition. An Eulerian decomposition D of H_p will correspond to a folded necklace if two conditions are satisfied. First, D has exactly one closed trail. Second, there is no crossing at any vertex. Given a vertex v of degree 4 in H_p with its edges in cyclic order e_1, e_2, e_3 and e_4, we say that D has a *crossing* at v if one of the closed trails of D uses e_3 after e_1. Then, the set of folded necklaces are a subset of the Eulerian decompositions of H_p with one closed trail and no crossings. We call this set of non-crossing Eulerian decompositions with one component the set of *folded-necklace configurations*. The following theorem tell us how many folded-necklace configurations there are.

Theorem 3.3. *The number of folded-necklace configurations equal the number of spanning trees of $L_{p/2}$.*

Proof. Let v a vertex of degree two in H_p, we can eliminate v by contracting one of its incident edges. If we continue this process, we will arrive at the graph $\hat{H}_p$ obtained from H_p by eliminating all the vertices of degree 2 in the described procedure. The graph $\hat{H}_p$ is a 2-connected 4-regular planar graph, then it is known that $\hat{H}_p$ is the medial graph of some graph, see chapter 2 of [18].

Consider the graph M with vertex set

$$V(M) = \{(i,j) \,|\, 0 \leq i,j \leq p,\, i \text{ and } j \text{ are odd}\},$$

and there is an edge between two vertices of $V(M)$ if they are at distance 2 in L_{p+1}. It is not difficult to check that $\hat{H}_p$ is the medial graph of M. Also, observe that M is isomorphic to $L_{p/2}$.

Now, any Eulerian partition of $\hat{H}_p$ corresponds to an Eulerian partition of H_p by splitting the vertices of degree 4 in the same way for both graphs. More important for us, the corresponding Eulerian partitions of H_p and $\hat{H}_p$ have the same number of components and either both have crossings or both have no-crossings.

We have the following general result [6, 14, 16, 17]. Given a graph Γ and Γ_m its medial graph, we have

$$T(\Gamma; x, x) = \sum_D (x-1)^{\gamma(D)-1},$$

where $T(\Gamma; x, y)$ is the Tutte polynomial of the graph Γ, the sum is over all Eulerian partitions of Γ_m such that each D has no crossings and $\gamma(D)$ denotes the number of closed trails of D. It is also known that $T(\Gamma; 1, 1)$ counts the number of spanning trees of Γ [6], thus, we have that the number of Eulerian partitions of Γ_m with one closed and no crossings equals the number of spanning trees of Γ. The theorem follows when we take Γ to be M. For the definition and properties of the Tutte polynomial, see [6]. $\square$

The previous theorem guarantee a bijection between folded-necklace configurations and spanning trees of the graph M. Explicit bijections have been constructed even for the general case of Γ and its medial graph Γ_m [13]. Here we describe a bijection for our case but the proof is just outline.

Proposition 3.4. *To each non-crossing Eulerian decomposition of H_p with one closed trail corresponds a spanning trees of $L_{p/2}$. Conversely, each spanning tree of $L_{p/2}$ defines a non-crossing Eulerian decomposition with one closed trail.*

Proof. For a vertex v of degree 4 in our planar drawing of H_p, its 4 incident edges e_1, e_2,e_3 and e_4, can be said to point out NE, NW, SW and SE respectively. A Eulerian decomposition D is said to split v horizontally if it uses e_1 and then e_2 and to split v vertically if it uses e_1 and then e_4.

Given an non-crossing Eulerian decomposition D of H_p with one component, we construct the following spanning subgraph T of M. The vertices (i, j) and $(i+2, j)$ of M with i and j odd are adjacent in T if and only if the vertex $(i+1, j)$ of H_p is split horizontally in D, for $1 \leq i \leq p-3$ and $1 \leq j \leq p-1$. Similarly, the vertices (i, j) and $(i, j+2)$ of M are adjacent in T if and only if the vertex $(i, j+1)$ is split vertically, for $1 \leq i \leq p-1$ and $1 \leq j \leq p-3$. Observe that the edge set of T is a subset of the edge set of M.

Any edge of T is surrounded by edges of the Eulerian trail, so if T had a cycle, this cycle would divide D in two components, contrary to our election of D. Also, each vertex of T is surrounded by the Eulerian trail D, and the edges of D do not cross any edge of M and thus of T. Therefore each connected component of T is surrounded by a closed trail of D, but as there is just one, T is connected. We conclude that T is a spanning tree of M.

For a given spanning tree T of M we construct the following Eulerian decomposition of H_p. Let (i, j) be a 4-degree vertex of H_p, then $0 < i, j < p$. As $i + j$ is odd, we consider two cases. First suppose that i is odd and j is even. In this case, if $(i, j-1)$ and $(i, j+1)$ are adjacent in T, we split (i, j) vertically, otherwise, we split it horizontally. In the case that i is even and j odd, if $(i-1, j)$ and $(i+1, j)$ are adjacent, we split (i, j) horizontally, otherwise we split it vertically.

Clearly, we obtain a non-crossing Eulerian decomposition D. But as any vertex of T is surrounded by the Eulerian trail, if D has more than 1 component, then T is not connected, contrary to our assumption that it is a spanning tree.

Finally observe that if we start with an Eulerian decomposition D, and we obtain a spanning tree T_D using the first construction, then when we apply the second construction to T_D we regain D. □

See Figure 4 for an example with H_6.

Corollary 3.5. *The number of folding necklaces of $\mathcal{NK}_{p^2}$ is at most the number of spanning trees of $L_{p/2}$.*

It has been studied the asymptotic behavior of several numerical invariants of the square lattice L_n. Of particular interest is the following result that is mentioned in [15]. The asymptotic behavior of the number of spanning trees of L_n, $t(L_n)$ is given by

$$\lim_{n\to\infty} t(n)^{1/n^2} = e^c \approx 3.209912556,$$

where c is the value of a double integral.

4 Two solution

4.1 A first solution

With the discussion of the previous section, we can now described easily a solution to the hinged dissection problem with k colours.

Given k colours we use a necklace of size p^2, $\mathcal{NK}_{p^2}$, with p even and at least $2k + 2\sqrt{k^2 - k}$. We choose k different segments of squares in $\mathcal{NK}_{p^2}$ of length $4(p-1)$ and we colour them with different colours. The remaining squares are colour arbitrarily. We fold the $\mathcal{NK}_{p^2}$ into a square in such a way that any of the chosen segments forms the perimeter of the square. The following spanning tree of $L_{p/2} = L_s$ describes, by using the bijection in Proposition 3.4, a folded-necklace configuration F that has this property. The tree T_s has edges joining the following pair of vertices:

$$\begin{array}{lll} (i,j),(i+1,j) & 1 \le i \le s-2, & 0 \le j \le s-1; \\ (0,j),(0,j+1) & 0 \le j \le s-2; & \\ (s-1,j),(s-1,j+1) & 0 \le j \le s-2; & \\ (0,s-1),(1,s-1). & & \end{array}$$

An example of such a tree for $p = 6$ is in Figure 4. The tree is in thick lines and the vertices are surrounded by circles.

A trivial observation is that if we can fold the necklace into a square, then we can unfold the square back to the necklace, and therefore, this problems are equivalent. So, it is just left to prove that F is actually a folded necklace.

To do this, we use the move called opening a corner of page 11 in Section 5. The move allow us to"open" the left bottom corner of F. For an example see Figure 4, on the left it appears a folded-necklace configuration with its

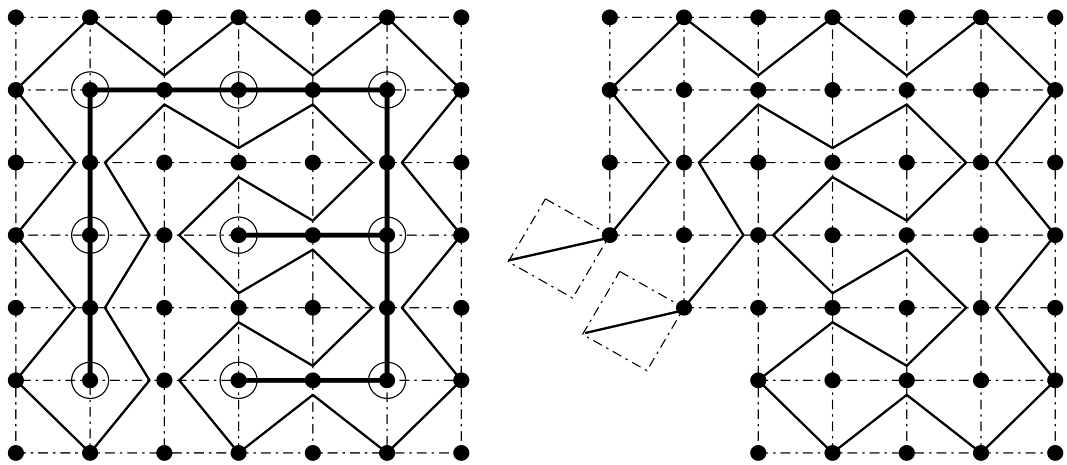

Figure 4: An example of an Eulerian partition and how to open a corner.

associated tree for $\mathcal{NK}_{36}$, and on the right, the folded-necklace configuration with the left bottom corner opened. Now, it is clear that the outer layer of squares can be unfolded. We can continue unfolding the left and upper part of the second outer layer and continue in zigzag with the rest of the layers. This process is depicted in Figure 5, on the left it appears the folded-necklace configuration of Figure 4 with the outer layer already unfolded, and on the right, the beginning of the zigzag process that will finish unfolding $\mathcal{NK}_{36}$.

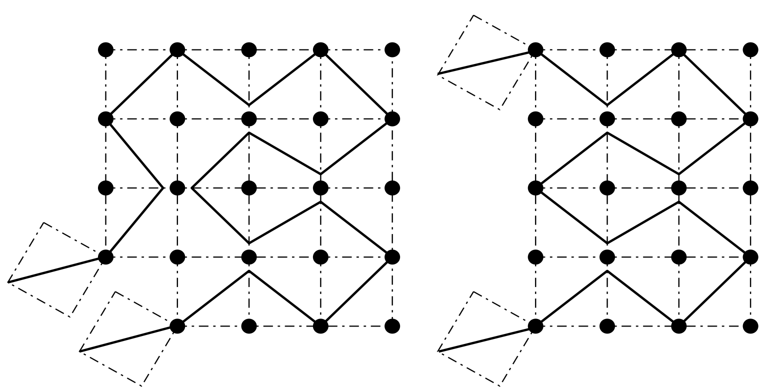

Figure 5: The process of unfolding in its last stages.

4.2 A second solution

The second solution for the hinge dissection problem with k colours uses the structure of chains of Section 2. It is very related to the solution of the previous subsection and the proof of its validity depends on the validity of the other solution, but we present it as we thought about it first and it reduces the number of pieces in the dissection by one half.

Take a chain $\mathcal{CH}_{p^2}$ and consider it as it were a necklace $\mathcal{NK}_{p^2}$. By the Subsection 4.1, if p is even and at least $2k + 2\sqrt{k^2 - k}$, we can used $\mathcal{CH}_{p^2}$ for a solution of the problem. But to reduce the number of pieces, we can choose $\lceil k/2 \rceil$ different segments of squares in $\mathcal{CH}_{p^2}$ of length $4(p-1)$. We now choose a pair of colours for each segment and we colour the outside part with one colour and the inside with the other colour. The remaining squares are coloured arbitrarily. As the inside of $\mathcal{CH}_{p^2}$, when consider as a necklace, can be change to be the outside, we reduce the number of segments required by a factor of $1/2$. So, this second solution requires a chain $\mathcal{CH}_{p^2}$, where p is even and at least $k + \sqrt{k^2 - k}$. Its validity follows form the validity in the case of necklaces.

5 Some allowable moves

In this section we show two moves, as explained in Section 3, that help us in the unfolding, if there is one, of a folded-necklace configuration. Here we use following trivial geometric fact. When a square is turn around a hinge in the bottom left corner h, the other three corners a, b and c, as in Figure 6, describe concentric circles. So, if there is no obstacles in the trajectory of a and b, there is no obstacle in the trajectory of c. In particular, a wall from h to c and from c to b is not an obstacle for the trajectory of c.

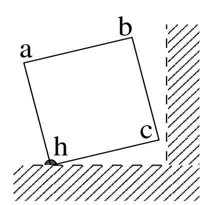

Figure 6: An square turning

We now described the move that is used in Subsection 4.1. Consider five squares $C_1, \ldots, C_5$ in the right hand corner of a folded-necklace configuration, where C_2 is on top of C_5, C_3 is on top of C_4 and C_2 is between C_1 and C_3. Suppose further that C_i is joint to C_{i+1} by a hinge, for $1 \leq i \leq 4$. Now, the following move is possible: turn C_1 and C_5 around their top left hinge by ϵ degrees at the same speed and simultaneously, also, at the same time, move C_2, C_3 and C_4 together, as one piece. This move is represented in Figure 7 and we call it *opening a corner*.

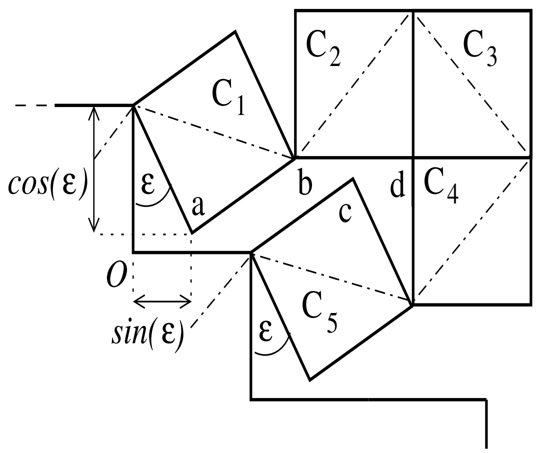

Figure 7: Opening a corner.

Lemma 5.1. *The move described above is possible and the squares $C_1, \ldots, C_5$ do not overlap.*

Proof. As we turn C_1 and C_5 at the same speed, the distance between their bottom right corners does not change, thus the L figure describe by C_2, C_3 and C_4 can be fit in the right position by an appropriate translation. If in Figure 7 we take the corner O to be the origin in the Cartesian plane, the corners a, b, c and d have the following coordinates:

$$\begin{aligned} a &= (\sin \epsilon, 1 - \cos \epsilon), \\ b &= (\sin \epsilon + \cos \epsilon, \sin \epsilon + 1 - \cos \epsilon), \\ c &= (\cos \epsilon + 1, \sin \epsilon), \\ d &= b + (1, 0). \end{aligned}$$

Thus, for any $0 \leq \epsilon \leq \pi$, there is not overlapping. □

The previous move is the only one that we need for Section 3. However, we consider interesting to give another move whose validity is not so intuitively obvious, in fact, a physical model, made of wooden squares and string, help us in the design of the move and in the proof of Lemma 5.2. Consider five squares $C_1, \ldots, C_5$ in the right hand corner of a folded-necklace configuration, where C_1,C_2 and C_3 are align horizontally next to each other and C_3,C_4 and C_5 are align vertically, each on top of the next. Suppose further that C_i is joint to C_{i+1} by a hinge, for $1 \leq i \leq 4$.

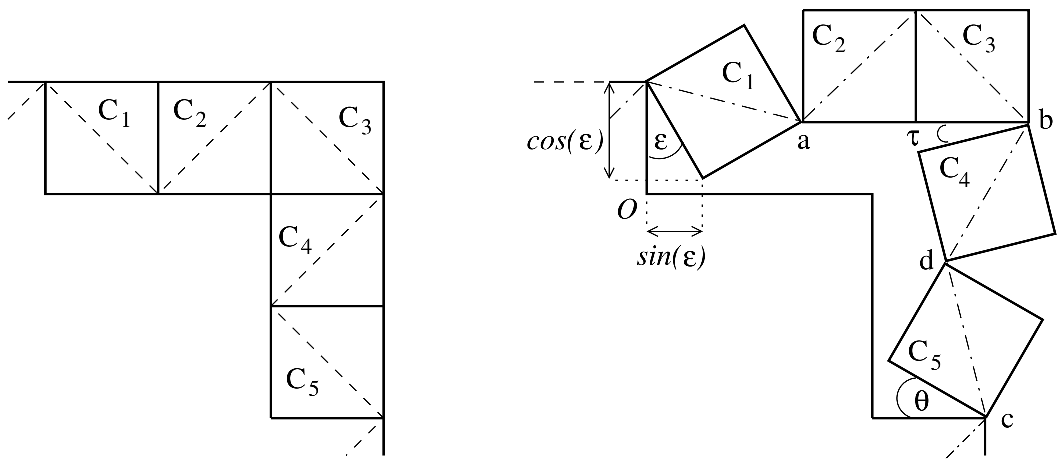

Figure 8: Another allowed move.

Lemma 5.2. *The move described in Figure 8 is possible and the squares $C_1, \ldots, C_5$ do not overlap.*

Proof. Suppose we turn C_1 around its top left hinge by ϵ degrees. As before, we take the cornet O in Figure 7 to be the origin in the Cartesian plane, then the corners a, b, c and have the following coordinates:

$$\begin{aligned} a &= (\sin\epsilon + \cos\epsilon, \sin\epsilon + 1 - \cos\epsilon), \\ b &= a + (2, 0), \\ c &= (3, -2). \end{aligned}$$

Thus, the distance from b to c is given by $2\sqrt{3+\sin\epsilon-2\cos\epsilon}$. For $0 \leq \epsilon \leq \arctan 3/4$, this distance is strictly less than $2\sqrt{2}$, so there exists a point d that satisfies that its distance to c equals its distance to b and both equals $\sqrt{2}$. Therefore, there exist angles θ and τ such that if we turn C_5 around its bottom left hinge by θ degrees and turn C_4 around its bottom right hinge by τ degrees, the squares C_3, C_4 and C_5 will stay joint by the respective hinges.

The observation made at the beginning of this section ensures that there is no overlapping. □

Observe that these two moves can "open" the corners of any folded-necklace configuration. The moves are local and other local moves can be designed to help in the process of unfolding a folded-necklace configuration. There are also global moves, for example the one that splits in 4 pieces a folding necklace, as depicted in Figure 9. In the picture, the polygons in thick line represent big regions of the folding necklace and the 4 dashed lines represent small sections of the Eulerian trail and the dots are 4 hinges in the necklace.

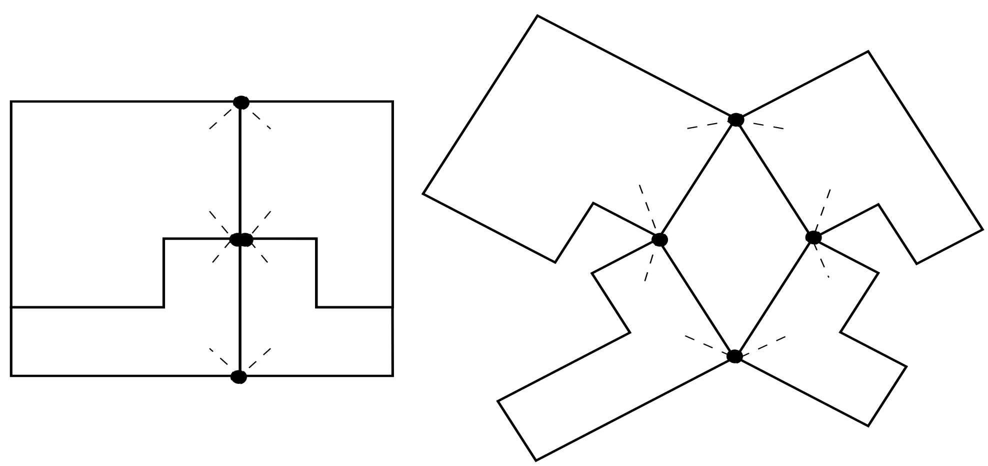

Figure 9: A global move that splits the folding necklace in 4 pieces.

6 Snakes

We have proved that a chain or a necklace can be folded into a square. Also, it is obvious that a chain can be folded into a line while necklaces do not have such a folding. So, a natural question is when a hinged structure of squares can be folded into a line. We called such a structure a *snake*. Two obvious requirements for a hinged structure T to be a snake are: first, T has to be connected (to be in one piece), and second, T cannot have cycles.

It is natural to called *tree-structure* a hinged structure of squares that is connected and with no cycles. Notice that checking if a given hinged structure with N squares is a tree-structure can be done in $O(N)$ steps.

A necessary condition for a tree-structure T to be a snake is that all of its squares have at most two hinges. However, not all of these tree-structures have the required folding as Figure 10 shows. Notice that this is a minimal size example as any tree-structure with 3 squares can be folded into a line.

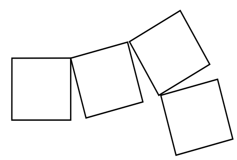

Figure 10: A tree-structure that cannot be folded into a line.

We present an algorithm to identify tree-structures with N squares that are snakes. The algorithm runs in time $O(N)$.

Let T be a tree-structure. First we check that each square in T has at most two hinges, if this is not the case, T is not a snake. Then, we find one square S with just one hinge (the existence of S is guaranteed by the fact that T is a tree-structure). These two procedures can be done in the same search and the time required is $O(N)$.

Given a square S and one of its corners, the other three corners are naturally named: left, right and front. Thus, for a square S and two hinges a and b in S, we can define a function $\varphi(S, a, b)$ which produces values in {L,R,F} such that $\varphi(S, A, b) =$ L (R, F, respectively) if the corner with hinge b is named left (right, front) according to the corner with hinge a.

Now, we do the following search on T that will produce a word over {L,R,F}. First, we write F in the output. Let S be the square we found in the previous step and a its only hinge. Then, find S' the only square joint to S and b the hinge in S' different from a. Write $\varphi(S, a, b)$ in the output and delete S and a. Continue the process with $S \leftarrow S'$ and $a \leftarrow b$. If S' has just one hinge (we arrive at the other end of T), write F in the output and terminate the procedure.

We can finish our algorithm by noticing that T is a snake if and only if the word produced by the previous procedure is in the language recognised by the non-deterministic finite automata described in Figure 11. In the automata, the state $\uparrow$ represents a hinge placed above the horizon and the state $\downarrow$ represents a hinge placed below the horizon, when trying to fold a tree-structure into an horizontal line from left to right. The symbol L (R, F respectively) represents a new square to be placed and which has a second hinge name left (right, front) according to the hinge already placed. We do not provided a proof of our statement , but the reader is invited to convince himself (herself).

Finally, the search and the simulation of the automata will take $O(N)$ steps, so the whole algorithm runs in time $O(N)$.

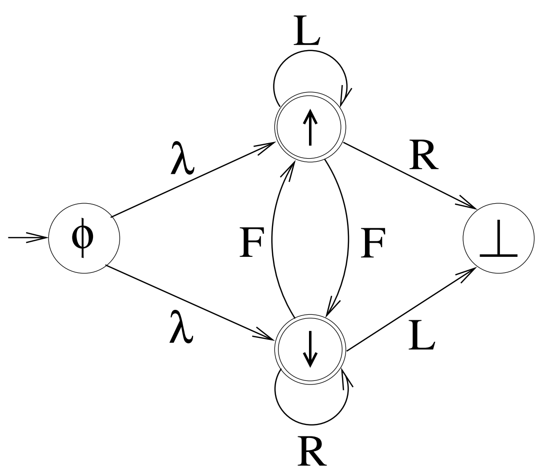

Figure 11: An automata that decides if a tree-structure is a snake.

7 Conclusion and open problems

We solve the hinge dissection problem with k colours. One solution uses a necklace $\mathcal{NK}_{p^2}$, with p an even number and at least $2k+2\sqrt{k^2-k}$. The other solution uses a chain $\mathcal{CH}_{p^2}$ with p an even number and at least $k+\sqrt{k^2-k}$.

The first natural question is when a folded-necklace configuration is a folded necklace, that is, when a folded-necklace configuration can be unfolded. The three moves mentioned in Section 5 have proved to be enough for all the folded-necklace configurations that we have tried. So, it is valid to ask if the set of folded-necklace configurations equals the set of folded necklaces of a given necklace.

A generalization of both problems is the following: consider as induced 2-connected subgraph G of the square lattice L_n, let v_0 be a vertex in G. Then, we can construct a graph H_G as the graph with vertex set the vertices of G at even distance from v_0 and two vertices adjacent if they are ad distance 2 in G. When G is L_n, H_G is precisely H_{n-1}. This graph is Eulerian, so there exists a non-crossing Eulerian decomposition with one closed trail. If we called these decompositions, folded-necklace configurations, the question is again when a given folded-necklace configuration is a folded necklace of sizes f, where f is the number of internal faces of G.

Another natural question is when a tree-structure can be folded into a square or in general to a rectangle.

It is worth mentioning the similar problems of straightening polygonal arcs and convexifying polygonal cycles. In both cases, not squares but line segments are considered and these pieces are, like in our case, joint by hinges. More precisely the questions are as follow: when a chain of line segments can be reconfigured to lie on a straight line and when a necklace of line segments can be reconfigured to form a planar convex polygon. The restrictions are as ours when consider the line segments as rigid bars. If a chain or necklace cannot be reconfigured, it is called locked. In [7] it is shown that no chain or necklace is locked. A different situation arises when the line segments are joint to form general trees and not just paths or cycles. In this case there exists trees that are locked, but for the precise result see [3, 8].

Also related to our work is [9] where, among other results, the authors consider hinged dissections of polyominoes by congruent right isosceles triangles. A polyomino is a finite collection of congruent squares such that the interior of their union is connected, and the intersection of two copies is either empty, a common vertex, or a common edge (notice that a square partitioned into p^2 smaller squares is a polyomino). It is shown in [9] that a chain of right isosceles triangles with $2n$ pieces can be folded into any polyomino with n squares. Another result is that 5 congruent squares cannot be hinged in such a way that they can be folded into all possible polyominoes of size 5.

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