On spanning trees and cycles of multicolored point sets with few intersections

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Abstract

Let $P_1, ..., P_k$ be a collection of disjoint point sets in $\Re^2$ in general position. We prove that for each $1 \leq i \leq k$ we can find a plane spanning tree T_i of P_i such that the edges of $T_1, ..., T_k$ intersect at most $kn(k-1)(n-k) + \frac{(k)(k-1)}{2}$, where n is the number of points in $P_1 \cup ... \cup P_k$. If the intersection of the convex hulls of $P_1, ..., P_k$ is non empty, we can find k spanning cycles such that their edges intersect at most $(k-1)n$ times, this bound is tight. We also prove that if P and Q are disjoint point sets in general position, then the minimum weight spanning trees of P and Q intersect at most $7n$ times, where $|P \cup Q| = n$ (the weight of an edge is its length).

1 Introduction

The study of geometric graphs, that is graphs whose vertex set is a collection of points on the plane in general position and its edges are straight line segments connecting pairs of vertices, has received a lot of attention lately. Numerous problems in which we want to draw graphs on the plane such that their vertices lie on the elements of a fixed point set have been studied. Ramsey type problems in which we want to color the edges or vertices of a geometric graph such that some specific forbidden subgraphs do not appear have also been studied. The interested reader may consult a recent survey by J. Pach [9] containing many results in this field. In this paper we are interested in studying problems of embeddings of geometric graphs on colored point sets. These problems have been studied for some time now, for example in [1, 2] the problem of embedding trees and alternating paths on bicolored point sets is studied. In [3, 4, 5] matching problems on colored point sets are studied. For two colored point sets we are interested in obtaining matchings in which every edge has its

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endpoints of different (or equal) color. A well known result states that given a colection P_{2n} of $2n$ points in general position, n blue, and n red, we can always match a blue with a red point in P_{2n} such that the line segments joining matched pairs of points do not intersect. For a recent survey dealing with numerous problems on colored point sets see [6]. Problems in which instead of coloring the vertices, we color the edges of geometric graphs, are studied in [7, 8].

Let P_n be a set with n points on the plane. A spanning tree of P_n is a connected geometric graph with vertex set P_n containing exactly $n-1$ edges. Let P and Q be disjoint point set. Tokunaga, studied and solved the problem of finding spanning trees for P and Q with the smallest possible number of edge intersections. It turns out that this number depends only in the order in which the elements of P and Q lying on the convex hull $Conv(P \cup Q)$ of $P \cup Q$ appear. More specifically, let $p_0, ..., p_{r-1}$ be the points on $Conv(P \cup Q)$ in clockwise order, and let i be the number of indexes j such that p_j and p_{j+1} are one in P and the other in Q, addition taken mod r. Then it is always possible to find spanning trees for P and Q such that their edges do not intersect if $i \leq 2$, otherwise they intersect exactly $\frac{i-2}{2}$ times. This implies for example that if all the points on the convex hull are the same color, or all the points in P which belong to the convex hull appear in consecutive order, then we can find spanning trees for P and Q which do not intersect, regardless of how many or where the remaining points of P and Q are.

In this paper we study the following problem: Let $P_1, ..., P_k$ be a family of disjoint point sets such that $P_1 \cup ... \cup P_k$ is in general position. Find spanning trees for $P_1, ..., P_k$ such that their edges have as few intersections as possible. In this paper we prove the following result:

Theorem 1 *Let $P_1, ..., P_k$ be a collection of disjoint point sets. Then we can find for each P_i a spanning tree T_i such that the total number of intersections among the edges of $T_1, ..., T_k$ is at most $(k-1)(n-k) + \frac{(k)(k-1)}{2}$ where $|P_1 \cup ... \cup P_k| = n$. This bound is tight within a factor of two from the optimal solution.*

We also give similar results for spanning cycles of families of point sets $P_1, ..., P_k$ in which $Conv(P_1) \cap ... \cap Conv(P_k)$ is non empty. Sharp bounds for this problem are obtained.

In the last section of this paper, we prove the following result that is of independent interest: Let P and Q be disjoint point sets, then their euclidean minimum weight spanning trees intersect at most $14n$ times. Using this we prove the following result: Let $P_1, \ldots, P_k$ be families of disjoint point sets such that $P_1 \cup ... \cup P_k$ is in general position. For each P_i let T_i be its euclidean minimum weight spanning tree, $i = 1, ..., k$. Then then the edges of these trees intersect at most $7kn$ times.

2 Spanning trees with few intersections

Given two disjoint point sets P_1 and P_2 it is not always possible to find spanning trees for them such that their edges do not intersect. In fact if we have $2s$ points which are the vertices of a convex polygon such that alternately they belong to P_1 and P_2, then it is easy to verify that any spanning tree for P_1 intersects any spanning tree for P_2 at least $s-1$ times, see Figure 1.

From here the following observation follows:

Observation 1 *There are families of point sets $P_1, ..., P_k$ with $|P_1 \cup, ..., \cup P_k| = sk$ such that their edges intersect at least $\frac{k(k-1)}{2}(s-1)$ times.*

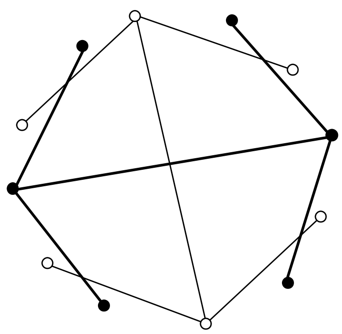

Figure 1: Two sets of points, each with six points such that any spanning tree of the set with solid points intersects any spanning tree for the remaining points at least five times.

If we consider a similar problem for three or more point sets our problem becomes much harder, even for points in convex position. Let $\mathcal{P}$ be a set of $n = ks$ points in convex position labelled $p_1, ..., p_{sk}$. Split $\mathcal{P}$ into k subsets $P_1, ..., P_k$ such that the element p_{i+rk} belongs to P_i, $r = 0, ..., s-1$. Finding for each P_i a spanning tree T_i, $1 \leq i \leq k$, such such that their edges have the fewest possible number of intersections is hard. We now show a set of spanning trees $T_1, ..., T_k$ such that their edges intersect at most $(\frac{3}{4}k^2 - k)(s-1) - \frac{k(k-2)}{4}$ times if k is even; otherwise they intersect $(\frac{3}{4}(k-1)^2 + \frac{k-1}{2})(s-1) - \frac{(k-1)^2}{4}$ times, i.e. the number of times their edges intersect is at most $\frac{3}{2}$ times the optimal solution.

For i even, let T_i be the tree containing the edges joining p_{i+ak} to p_{i+bk}, $a+b = s+1$ or $a+b = s+2$, $1 \leq a, b \leq s$. For i odd, T_i, is the tree containing the edges joining p_{i+ak} to p_{i+bk}, $a+b = s$ or $a+b = s+1$, $1 \leq a, b \leq s$. Notice that two trees T_i and T_j intersect $s-1$ times if i and j have different parity; otherwise they intersect $2(s-1)-1 = 2s-3$ times. See Figure 2. Therefore these trees intersect exactly $(\frac{3}{4}k^2 - k)(s-1) - \frac{k(k-2)}{4}$ if k is even; and $(\frac{3}{4}(k)^2 + \frac{k-1}{2})(s-1) - \frac{(k-1)^2}{4}$ if k is odd. Moreover we believe that this configuration is, in fact, close to the optimal solution for point sets in convex position.

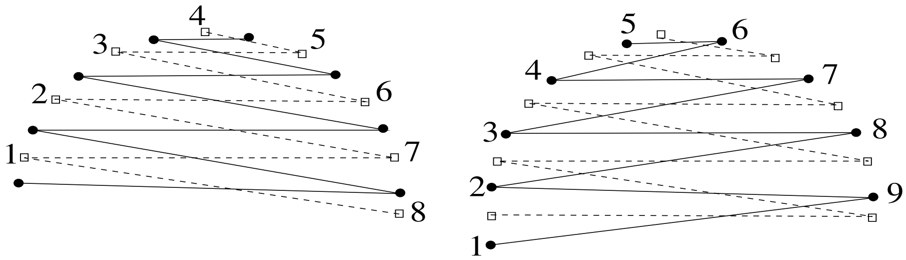

Figure 2: On the left hand side we have T_i, i odd, with dashed lines and T_j, j even, with solid lines; $s = 8$ so, they intersect 7 times. On the right hand side we have T_i and T_j, i, j even; as $s = 9$, they intersect 15 times.

We proceed now to study our problem for point sets in general position. Suppose w.l.o.g. that the points in $P_1 \cup ... \cup P_k$ have different x-coordinates, and $|P_i| \geq 2$, $i = 1, ..., k$. Assume that for every i the elements of P_i are labeled $p_{i,1}, p_{i,2}, \ldots, p_{i,r_i}$ such that if $r < s$ then the x-coordinate of $p_{i,r}$ is smaller than the x-coordinate of $p_{i,s}$. Let T_i be the path with vertex set P_i in which $p_{i,j}$ is connected to $p_{i,j+1}$ by an edge denoted by $e_{i,j}$, $j = 1, \ldots, r_i - 1$. See Figure 3.

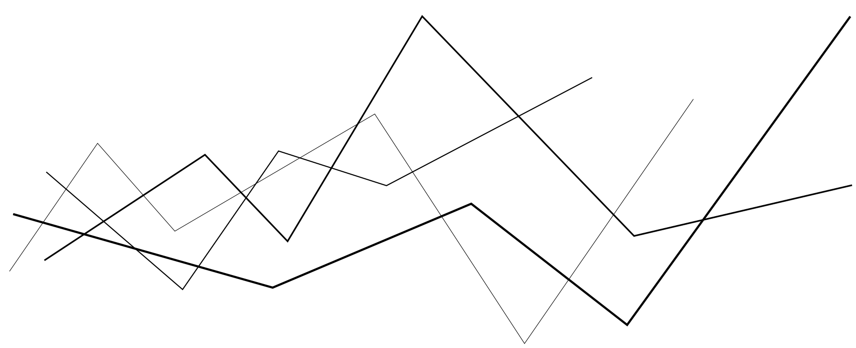

Figure 3: A colection of four point sets and their spanning trees. The point sets are the vertices of our trees, which turn out to be paths.

Lemma 1 *The edges of T_i and T_j intersect at most $r_i + r_j - 3$ times.*

Proof: Our result is clearly true if $r_i + r_j \leq 4$, or one of T_i or T_j has exactly one edge. Suppose now that the x coordinate of $p_{i,2}$ is smaller than that of $p_{j,2}$. Then the edge $e_{i,1}$ of T_i joining $p_{i,1}$ to $p_{i,2}$ intersects at most one edge of T_j, namely the edge $e_{j,1}$ joining $p_{j,1}$ to $p_{j,2}$. Remove $p_{i,1}$ from p_i, and by induction our result follows. ∎

In a similar way we can prove:

Lemma 2 *The edges of* $T_1, \ldots, T_k$ *intersect at most* $(k-1)(n-k) + \frac{(k)(k-1)}{2}$ *times, where* $|P_1 \cup ... \cup P_k| = n$

Proof: Our result is true if $T_1, ..., T_k$ have together at most k edges, in fact in this case if all of them intersect each other, their total number of intersections is $\frac{(k)(k-1)}{2}$. Suppose then that our trees contain more than k edges, and let $e_{i,1}$ be such that the x-coordinate of $p_{i,2}$ is smaller than the x-coordinate of $p_{j,2}$, $i \neq j$, $1 \leq j \leq k$. Then the edge $e_{i,1}$ joining $p_{i,1}$ to $p_{i,2}$ intersects at most $k-1$ edges, i.e. in each T_j $e_{i,1}$ intersects at most the edge $e_{j,1}$ joining $p_{j,1}$ to $p_{j,2}$, $j \neq i$. Removing this edge, and p_{i_1} from P_i, and proceeding by induction on $P_1, \ldots, P_i - \{p_{i,1}\}, \ldots, P_k$ our result follows. ∎

Observe that the bound determined in Lemma2 is within a factor of two of that in Observation 1. Theorem 1 follows directly from Observation 1 and Lemma 2.

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3 Spanning Cycles

We now study the following problem: Let $P_1, ..., P_k$ be a family of disjoint point sets such that $Conv(P_1) \cap, ..., \cap Conv(P_k) \neq \emptyset$. Find a family of spanning cycles $C_i, ..., C_k$ of $P_1, ..., P_k$ respectively with few intersections. We prove:

Theorem 2 *Let* $P_1, ..., P_k$ *be a family of disjoint point set such that* $Conv(P_1) \cap, ..., \cap Conv(P_k) \neq \emptyset$. *Then for each* P_i *we can find a cycle* C_k *which covers the vertices of* P_i *such that the edges of all cycles* $C_i, ..., C_k$ *intersect at most* $(k-1)n$ *times. Our bound is optimal.*

Proof: Let q be a point in the interior of $Conv(P_1) \cap, ..., \cap Conv(P_k)$. For each P_i define a cycle C_i^q as follows: Sort the elements of P_i around q in the counterclockwise order according to their slope and label them $p_{i,1}, ..., p_{i,r_i}$ (see Figure 4(a)).

A straightforward modification to our counting argument in Lemma 2 shows that the edges of $C_1, ..., C_k$ intersect at most $(k-1)n$ times. To show that our bound is tight, choose $n = kr$, and choose kr points on a unit circle labeled $p_1, ..., p_{kr}$, and let $P_i = \{p_{i+ks} : k = 0, ..., r-1\}$. It is easy to see that the (unique) cycles C_i that cover the vertices of each P_i, $i = 1, ..., k$ intersect $(k-1)(kr) = (k-1)n$ times, see Figure 4(b). ∎

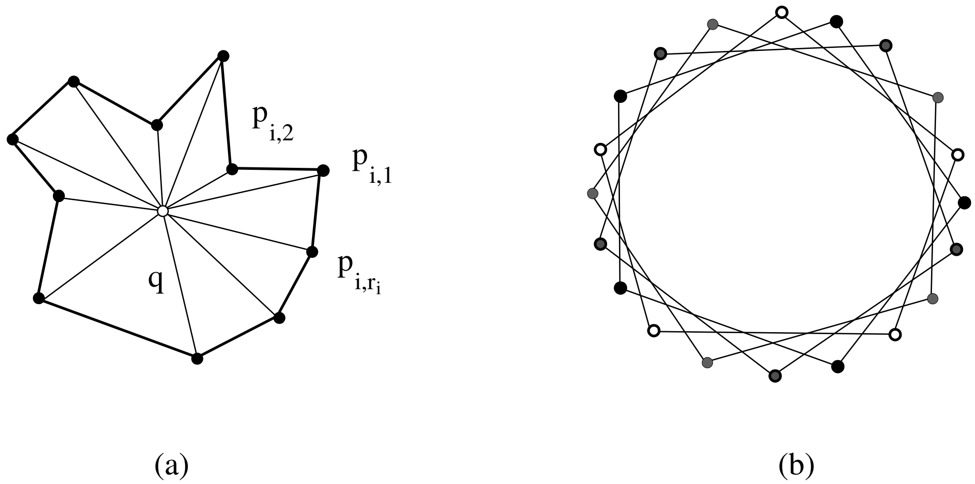

Figure 4:

4 Minimum weight spanning trees

The euclidean minimum weight spanning tree of a point set P_n is a tree with vertex set P_n such that the sum of the lengths of its edges is minimized. In this section we prove that if $P_1, ..., P_k$ are disjoint point sets and $T_1, ..., T_k$ are their corresponding euclidean minimum weight spanning trees then the total number of intersections among their edges is at most $7(k-1)(n-k)$ where $|P_1 \cup ... \cup P_k| = n$. Our proof is based on the following observation that is easy to prove: Let T_i and T_j be the minimum weight spanning trees of P_i and P_j. Let e be any edge of T_i. Then there is a constant c such that e intersects at most c edges of T_j whose length is grater than or equal to the length of e. It follows that the edges of T_i and T_j intersect a linear number of times. In fact, we can prove that c is at most 9, however the proof is long, tedious, and unenlightening. We skip the details, they can be supplied by the authors upon request. Summarizing we have:

Lemma 3 *Let T_1 and T_2 be the minimum weight spanning trees of two point sets P_1 and P_2, and e any edge of T_1. Then e intersects at most 9 edges of T_2 whose length is greather than or equal to the lenght of e.*

We now prove:

Theorem 3 *Let $T_1, ..., T_k$ be respectively the minimum weight spanning trees of k point sets $P_1, ..., P_k$ such that $|P_1 \cup ... \cup P_k| = n$. Then the edges of $T_1, ..., T_k$ intersect at most $9(k-1)(n-k)$ times.*

Proof: Observe first that since $|P_1 \cup ... \cup P_k| = n$, $T_1, ..., T_k$ have exactly $n-k$ edges. Let us construct the intersection graph H of the set of edges of $T_1, ..., T_k$, that is the graph whose vertex set is the set of all edges of $T_1, ..., T_k$, two of which are adjacent if they intersect. Orient the edges of this graph as follows: If two edges $e \in T_i$ and $e \in T_j'$ intesect and e is longer than e' orient the edge in H joining them from e to e', see Figure 5.

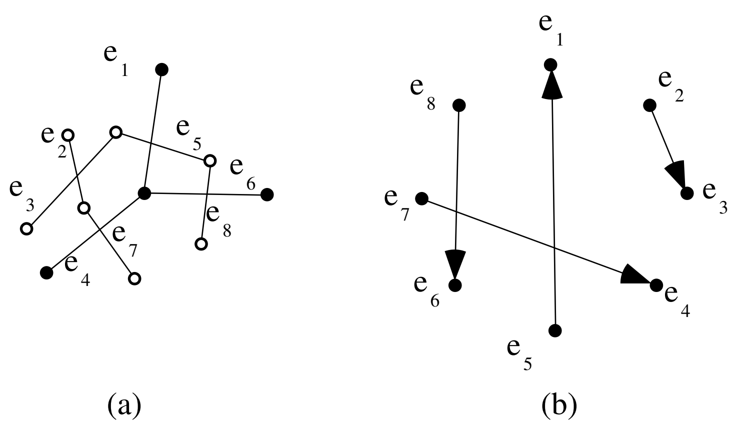

Figure 5: The intersection graph of three minimum weight spanning trees.

By Lemma 6 every edge in T_i intersects at most 9 edges in each T_j, $i \neq j$ which are the same lenght or longer than itself. Thus the out-degree of each vertex of H is at most $9(k-1)$. Our result follows. ■

Observe that for the case when we have two point sets P and Q such that $|P \cup Q| = n$, our previous results implies that the edges of their minimum weight spanning trees intersect at most $9(n-2)$ times. This bound is far from optimal. In fact we have been unable to produce examples in which the minimum weight spanning trees of P and Q intersect more than $2n-4$ times. An example is constructed as follows: P consists of 3 points r, s, t such that r and s are equidistant from t, and the angle $\angle rts$ is slightly bigger than $\frac{\pi}{3}$. The points of Q lie on a zig-zag polygonal such that each second segment of it is parallel, and the angle between two consecutive segments is $\frac{\pi}{2}$ as shown in Figure 6.

We conclude by posing the following question:

Open problem 1 *Is it true that the edges of the minimum weight spanning trees of any two point sets P and Q such that $|P \cup Q| = n$ intersect at most $2n - c$ times, c a constant?*

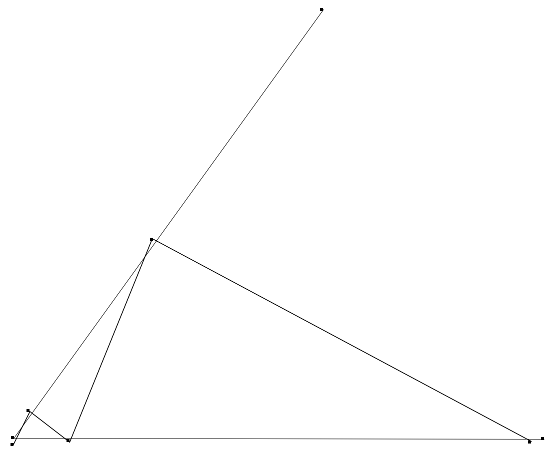

Figure 6: P has 3 points, and Q 5. The number of elements of Q can be increased to $n-3$, $n \geq 4$. The number of edge intersections of the minimum weight spanning trees of P and Q is $2n-4$.

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