Milan j. math. *Online First*

DOI 10.1007/s00032-008-0088-x

Milan Journal of Mathematics

Motivic Homotopy Theory

Marc Levine

Abstract. We give an informal discussion of the roots and accomplishments of motivic homotopy theory.

Mathematics Subject Classification (2000). 14F45, 55P43.

Keywords. Motivic homotopy theory, motivic cohomology, algebraic K-theory, algebraic cobordism.

1. Introduction

Algebraic geometry and topology share a long history of interaction, cross-fertilization and competition. The latest phase involves the newly created field of *motivic homotopy theory*. This can be thought of as an expansion of homotopy theory to a setting that directly involves algebraic geometry, and has enabled the introduction of techniques of algebraic topology to problems in algebra, number theory and algebraic geometry. We will discuss the sources of this development together with a look at the recent applications of the theory. For further reading, we recommend the texts [2, 3, 10, 27, 33, 56].

2. Two parallel worlds

Algebraic topology and algebraic geometry deal with rather different objects, but often borrow methods and approaches from one another. Here is a very rough dictionary listing some of the parallels:

The author gratefully acknowledges the support of the Humboldt Foundation and the ICTP, as well as the support of the NSF via the grant DMS-0457195.

Algebraic topology		**Algebraic geometry**
singular (co)chains	$\leftrightarrow$	algebraic cycles
cup product	$\leftrightarrow$	intersection theory
homology	$\leftrightarrow$	adequate equivalence relations
singular cohomology	$\leftrightarrow$	motivic cohomology
topological K-theory	$\leftrightarrow$	algebraic K-theory

In the last 15 years or so, constructions of Morel-Voevodsky [36], Morel [35, 33], Voevodsky [55], Jardine [16] and others have given a framework to make the rough parallel more precise. The Morel-Voevodsky theory fills in the dictionary:

Algebraic topology		**Algebraic geometry**
homotopy theory	$\leftrightarrow$	motivic homotopy theory
spaces	$\leftrightarrow$	algebraic varieties + ...
$[0,1]$	$\leftrightarrow$	$\mathbb{A}^1$
S^1	$\leftrightarrow$	$\mathbb{A}^1/\{0,1\}$ and $\mathbb{A}^1 \setminus \{0\}$
generalized cohomology	$\leftrightarrow$	generalized cohomology
of space: spectra		of varieties: T-spectra
complex cobordism	$\leftrightarrow$	algebraic cobordism
homotopy groups of spheres	$\leftrightarrow$	Witt groups +?

Still, the most studied part of this parallel is in the area of singular cohomology. One can make a tour through the singular cohomology of topological spaces, starting with the most concrete and steadily increasing the level of abstraction. One begins with a vague notion of geometric "cycles" on a manifold, refining this to the complex of singular (co)chains, and then generalizing to abstract sheaf cohomology. Finally, one gives this all a categorical framework by introducing the derived category of sheaves. Via the derived push-forward functor to a point, one can recover cohomology (with constant coefficients) on some X as the morphisms in the derived category $D(\mathbf{Ab})$ with source the singular chain complex $C_*(X)$.

The parallels in algebraic geometry are

Algebraic topology		**Algebraic geometry**
geometric cycles	$\leftrightarrow$	algebraic cycles and intersection theory
singular (co)homology	$\leftrightarrow$	Suslin homology and motivic cohomology
sheaf theory	$\leftrightarrow$	sheaves with transfer
$D(\mathbf{Ab})$ and the functor $X \mapsto C_*(X)$	$\leftrightarrow$	$DM(k)$ and the functor $X \mapsto C^{\mathrm{Sus}}(X)$

3. Cohomology in topology and algebraic geometry

Let's first discuss the parallel worlds of cohomology.

3.1. Algebraic topology

Algebraic topology began with the introduction of homology and the fundamental group. Basic constructions include:

• In many ways, the *path integral* started the ball rolling. This was generalized to allow integrals over geometric figures of various dimensions.

• A good theory of homology needs the notion of the singular chain complex $C_*(X)$ of a CW complex X. This in turn relies on the n-*simplex*

$$\Delta_n := \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum_i t_i = 1, t_i \geq 0\}.$$

These fit together via the coface and codegeneracy maps

$$\begin{aligned} &\delta_i^n : \Delta_n \to \Delta_{n+1} \\ &(t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n) \end{aligned}$$

$$\begin{aligned} &s_i^n : \Delta_n \to \Delta_{n-1} \\ &(t_0, \ldots, t_n) \mapsto (t_0, \ldots, t_{i-1} + t_i, t_{i+1}, \ldots, t_n). \end{aligned}$$

The collection of spaces $\{\Delta_n\}$ with the coface maps δ_i^n and codegeneracy maps s_i^n is a particular example of a *cosimplicial space*. The group of singular chains on X of dimension n, $C_n(X)$, is just the free abelian group on all continuous maps

$$\sigma : \Delta_n \to X.$$

Make the $C_n(X)$ into a complex by defining

$$d_{n-1}\sigma := \sum_{i=0}^{n}(-1)^i\sigma \circ \delta_i^{n-1}.$$

The *singular homology* of X is

$$H_n(X,\mathbb{Z}) := H_n(C_*(X)).$$

Cohomology is defined by using the complex of duals of $C_n(X)$:

$$C^n(X,A) := \mathrm{Hom}(C_n(X),A)$$

with differential $d^n : C^n(X,A) \to C^{n+1}(X,A)$ the dual of d_n. Cohomology with values in a commutative ring A has a ring structure, the *cup product.*

For a compact manifold M, one can use Poincaré duality to transfer the cup product on cohomology to the *intersection product* on homology. Lefschetz described how to make this product directly on homology chains by moving singular chains into good position.

• For reasonable spaces, the singular cohomology can be given by Čech cohomology, which in turn is directly related to sheaf cohomology. The introduction of sheaf cohomology and the allied theory of homological algebra lead to a much better conceptual understanding of cohomology theories, giving for example a very clear explanation of the de Rham theorem identifying singular cohomology with de Rham cohomology.

• A systematic development of sheaf cohomology and homological algebra leads to the *derived category* of sheaves and the whole theory of triangulated categories. This in turn gave generalizations of many important constructions, for instance, the generalization of the Serre spectral sequence of a fibration to the Leray spectral sequence for the cohomology of a sheaf.

3.2. Algebraic geometry

Let us now give a discussion of parallel developments in algebraic geometry

• *Algebraic cycles and intersection theory.* Very early on, algebraic geometers were interested in giving a purely geometric version of the intersection product in homology. They used the geometric approach of Lefschetz to define an intersection product on *algebraic cycles* on a smooth projective variety X.

Definition 3.2.1. An algebraic cycle Z of dimension d on a variety X (over a field k) is a formal sum

$$Z := \sum_i n_i Z_i$$

with the $n_i \in \mathbb{Z}$ and the Z_i integral subschemes of X of dimension d over k. Let $z_d(X)$ be the set of all such, this is a group under componentwise addition. For $Z := \sum_i n_i Z_i$, set $|Z| := \cup_i Z_i$.

If X has dimension d_X, set

$$z^d(X) := z_{d_X - d}(X).$$

By various means, one can define the intersection of two cycles Z, W on a smooth variety X, assuming the intersection had the proper dimension, i.e., for $Z \in z^p(X)$, $W \in z^q(X)$ each component of $|Z| \cap |W|$ should have codimension $p+q$ on X. This gives the partially defined intersection product

$$Z \cdot W := \sum_T m(T; Z, W)T$$

where T runs over the components of $|Z| \cap |W|$ and the $m(T; Z, W)$ are the intersection multiplicities.

Roughly speaking the intersection multiplicity $m(T; Z, W)$ is a measure of how tangent W and Z are at the generic point of T. For instance, if W and Z are manifolds along an open subset U of T and intersect transversely along U, then $m(T; Z, W) = 1$. The most direct method of defining the multiplicities was found by Serre [42], who gave a formula using homological algebra.

Intersection theory is interesting to algebraic geometers since many geometric problems can be phrased as the computation of an intersection multiplicity. For instance: the number of lines in the plane that are simultaneously tangent to a curve D of degree d and a curve E of degree e is the intersection degree of the dual curves D' and E' to D and E, if one counts the lines with the correct multiplicity. The degrees of D' and E' are $d' = d(d-1)$, $e' = e(e-1)$ and Bezout's theorem shows that the intersection number of D' and E' is $ed(e-1)(d-1)$.

• *Cycles and (co)homology.* The idea that algebraic cycles are somehow like singular cycles is an old one, but there is a problem in this analogy: What replaces the singular *chains*? There is an evident algebraic analog of the singular simplex Δ_n, the *algebraic n-simplex* Δ^n_k, defined as the hyperplane in affine $n+1$ space over k given by the equation

$$t_0 + \ldots + t_n = 1.$$

The co-face and co-degeneracy maps for Δ_n are all linear maps in the barycentric coordinates t_i, so they have a direct extension to maps among the Δ^n_k. Formally speaking, this makes the assignment $n \mapsto \Delta^n_k$ into a *cosimplicial scheme*. The coordinates $t_1, \ldots, t_n$ give an isomorphism of Δ^n_k with the affine n-space $\mathbb{A}^n_k$.

For a k-variety X, one could just extend the singular chain complex construction by taking $C^{\mathrm{alg}}_n(X)$ to be the free abelian group on the set of algebraic maps $\Delta^n_k \to X$. However, for most X, there are no non-constant maps $\mathbb{A}^n \to X$, so this theory would be rather uninteresting.

Andrei Suslin [44] came up with the correct definition, using the Dold-Thom theorem as a guide. This result states that the reduced homology groups of a pointed CW complex X are the same as the homotopy groups of the infinite symmetric product of X:

$$\tilde{H}_n(X, \mathbb{Z}) \cong \pi_n(\mathrm{Sym}^\infty X).$$

Suslin defined the algebraic analogy of maps of S^n to $\mathrm{Sym}^\infty X$ as the group of *finite cycles*:

$$\begin{aligned} C^{\mathrm{Sus}}_n(X) := \mathbb{Z}\{W \subset \Delta^n \times X \;|W &\text{ is irreducible and} \\ &W \to \Delta^n_k \text{ is finite and surjective}\}. \end{aligned}$$

Such a W gives formally a map $\Delta^n_k \to S^d(X)$ by sending $t \in \Delta^n_k$ to the cycle $W \cdot (t \times X)$ on X, where we view

$$W \cdot (t \times X) = w_1(t) + \ldots + w_d(t)$$

as a sum of d points of X, d being the degree of W over X.

Pull-back by the alternating sum of the co-face maps for Δ^*_k defines a differential for $C^{\mathrm{Sus}}_*(X)$; the resulting homology is the *Suslin homology* of X:

$$H^{\mathrm{Sus}}_n(X) := H_n(C^{\mathrm{Sus}}_*(X)).$$

What about cohomology? Here the situation exposes a crucial difference from the topological case, in that there are *two* types of S^1s that naturally occur in algebraic geometry. In topology, one has "wrong-way" or Gysin maps on cohomology for a closed codimension d immersion of oriented manifolds $i : V \to W$. The Gysin map is constructed by identifying a tubular neighborhood T of V in W with the normal bundle $N_{V/W}$, and using the Thom isomorphism

$$H^{n+d}(N_{V/W}/(N_{V/W} \setminus 0_V)) \cong H^n(V).$$

Excision gives us an isomorphism

$$H^{n+d}(N_{V/W}/(N_{V/W} \setminus 0_V)) \cong H^{n+d}(T/(T \setminus V)) \cong H^{n+d}(W/(W \setminus V))$$

and pull-back by the quotient map $W \to W/(W \setminus V)$ gives the natural map

$$H^{n+d}(W/(W \setminus V)) \to H^{n+d}(W).$$

Putting these all together gives the Gysin map

$$i_* : H^n(V) \to H^{n+d}(W).$$

One can perform an analogous construction in algebraic geometry with the help of the Morel-Voevodsky category of *spaces over* k, $\mathbf{Spc}(k)$, and the motivic unstable homotopy category $\mathcal{H}(k)$ (see [36]). More about these constructions later; for the purpose of our discussion, we want to concentrate on a crucial difference between the Thom space construction in topology and in algebraic geometry. In the case of V a point v of W, the normal bundle is just the tangent space $T_{W,v}$ and the Thom space is homotopy equivalent to the quotient of real projective spaces

$$T_{W,v}/(T_{W,v} \setminus 0) \sim \mathbb{RP}(T_{W,v} \oplus \mathbb{R})/\mathbb{RP}(T_{W,v}) \sim S^d$$

where d is the dimension of W. If now V and W are algebraic varieties, we replace the real projective spaces with the scheme-versions:

$$T_{W,v}/(T_{W,v} \setminus 0) \sim \mathbb{P}(T_{W,v} \oplus e)/\mathbb{P}(T_{W,v})$$

where $e \to W$ is the trivial rank one bundle. This latter quotient is *not* the same as a topological sphere in $\mathcal{H}(k)$. In fact, if we take the simplest case of dimension one, we just get $\mathbb{P}^1$, pointed by ∞. Noting that $\mathbb{P}^1$ is a union of two $\mathbb{A}^1$s over their intersection $\mathbb{G}_m := \mathbb{A}^1 \setminus 0$, and that $\mathbb{A}^1$ is contractible in the Morel-Voevodsky homotopy category, it follows that $\mathbb{P}^1$ is isomorphic in $\mathcal{H}(k)$ to the suspension of $\mathbb{G}_m$, i.e.,

$$\mathbb{P}^1 \sim S^1 \wedge \mathbb{G}_m,$$

while $S^2 \sim S^1 \wedge S^1$. The space S^2, $\mathbb{P}^1$ are *not* homotopy equivalent, so one needs to keep track of how many "Tate circles" $\mathbb{G}_m$ go into the production of a sphere-like object. This number is called the "weight" of the algebro-geometric sphere, and we use the notation $S^{a,b}$ to denote the sphere $(S^1)^{\wedge a-b} \wedge (\mathbb{G}_m)^{\wedge b}$. For example, the sphere $\mathbb{P}^d/\mathbb{P}^{d-1}$ that occurs in the algebraic Thom space is an $S^{2d,d}$.

Once we have noted this crucial difference, we can follow the topological lead in defining the algebraic version of cohomology, known as *motivic cohomology*. The Dold-Thom theorem [9] tells us that the infinite symmetric product $\mathrm{Sym}^\infty S^d$ is an Eilenberg-MacLane space $K(\mathbb{Z}, d)$, i.e., has exactly one non-zero homotopy group $\pi_d = \mathbb{Z}$. By obstruction theory, one can compute cohomology by

$$H^n(X) = \pi_{d-n}(\mathrm{Maps}(X, \mathrm{Sym}^\infty(S^d)))$$

for $n \leq d$. Replacing S^d with $S^{2d,d}$ gives us

$$H^{p,d}(X, \mathbb{Z}) := \pi_{2d-p}(\mathrm{Maps}(X, \mathrm{Sym}^\infty(\mathbb{P}^d/\mathbb{P}^{d-1})).$$

Now we need to make sense of the mapping space, using Suslin's construction as a guide. To define $\pi_n(\mathrm{Maps}(X, \mathrm{Sym}^\infty(Y))$, let

$$\begin{aligned} C_n^{\mathrm{Sus}}(Y)(X) := \mathbb{Z}\{W \subset \Delta^n \times X \times Y \ |W \text{ is irreducible, and}\\ W \to \Delta^n \times X \text{ is finite and surjective}\\ \text{onto some component of } \Delta^n \times X\}. \end{aligned}$$

The $C_n^{\mathrm{Sus}}(Y)(X)$ form a complex $C_*^{\mathrm{Sus}}(Y)(X)$, which is contravariantly functorial in X and covariant in Y. For a closed immersion $Z \to Y$, define

$$C_n^{\mathrm{Sus}}(Y/Z)(X) := C_n^{\mathrm{Sus}}(Y)(X)/C_n^{\mathrm{Sus}}(Z)(X).$$

The homology of $C_*^{\mathrm{Sus}}(Y/Z)(X)$ is a good replacement of the homotopy groups of the "mapping space" $\mathrm{Maps}(X, \mathrm{Sym}^\infty(Y/Z))$, which leads to

Definition 3.2.2. Let X be a smooth variety over X. The *weight d motivic cohomology* of X is

$$H^{p,d}(X, \mathbb{Z}) := H_{2d-p}(C_*^{\mathrm{Sus}}(\mathbb{P}^d/\mathbb{P}^{d-1})(X)).$$

One often sees the notation $H^p(X, \mathbb{Z}(d))$ for $H^{p,d}(X, \mathbb{Z})$.

Remark 3.2.3. The analogy suggests that our definition of $H^{p,d}(X, \mathbb{Z})$ should only be valid for $p \leq 2d$, and that one needs to extend the definition for larger p by a stabilization process, i.e.,

$$H^{p,d}(X, \mathbb{Z}) := \lim_{N\to\infty} H^{p+2N,d+N}(X \wedge (\mathbb{P}^N/\mathbb{P}^{N-1}), \mathbb{Z}).$$

In fact, the limit is constant, equal to its value at $N = 0$, so we do have the correct definition. The fact that the limit is constant follows from Voevodsky's cancellation theorem [57, Chapter 5, Theorem 4.3.1].

• *Sheaves and the derived category.* There is an algebraic version of the derived category $D(\mathbf{Ab})$, which contains the Suslin complex of algebraic varieties, or more generally, the complexes computing weight n motivic cohomology. This is Voevodsky's *triangulated category of motives over k*, $DM_-^{\mathrm{eff}}(k)$ (see [57]). The objects of $D(\mathbf{Ab})$ are just complexes of abelian groups, and morphisms are given by starting with the category of chain homotopy classes of maps of complexes and then inverting the *quasi-isomorphisms*, i.e., the maps $f : C \to C'$ which induce an isomorphism on all cohomology groups. $DM_-^{\mathrm{eff}}(k)$ is defined similarly, but one needs to incorporate the sheaf theory of smooth algebraic varieties over k

To do this, and have at the same time a relation with the Suslin complex construction, one must first enlarge the category of smooth algebraic k-scheme, $\mathbf{Sm}/k$, to the *category of finite correspondences* $SmCor(k)$. The objects are the same, but one defines

$$\mathrm{Hom}_{SmCor(k)}(X,Y) := C_0^{\mathrm{Sus}}(Y)(X)$$

i.e., the group of cycles on $X \times Y$ which are finite over X, as explained above. If we think of a finite cycle $W \subset X \times Y$ as a map $X \to S^d(Y)$, then we are led to a good idea of the composition law. One can define the composition of correspondences purely in terms of intersection theory by

$$W' \circ W := p_{XZ*}(W \times Z \cdot X \times W')$$

for $W \subset X \times Y$, $W' \subset Y \times Z$, with the intersection taking place on $X \times Y \times Z$. Sending a map $f : X \to Y$ to its graph $\Gamma_f \subset X \times Y$ gives the inclusion functor

$$i : \mathbf{Sm}/k \to SmCor(k),$$

which shows how $SmCor(k)$ is an enlargement of $\mathbf{Sm}/k$.

A presheaf (of abelian groups) on $\mathbf{Sm}/k$ is just a functor

$$P : \mathbf{Sm}/k^{\mathrm{op}} \to \mathbf{Ab};$$

similarly, define a *presheaf with transfers on* $\mathbf{Sm}/k$ to be a presheaf on $SmCor(k)$, that is, a functor $P : SmCor(k)^{\mathrm{op}} \to \mathbf{Ab}$. We write $\mathrm{PST}(k)$ for the category of presheaves with transfers on $\mathbf{Sm}/k$.

For sheaves, one needs a topology. We won't go into the whole story of Grothendieck topologies here, except to say that, roughly speaking, one can generalize standard point-set topology by replacing open subsets $U \subset X$ with maps $U \to X$, and intersection $U \cap V$ of open subsets with fiber product $U \times_X V$. In this way, one can define a Grothendieck topology on $X \in \mathbf{Sm}/k$ by selecting an appropriate collection of maps, called *covering families*, which need to satisfy certain axioms (see [1, 29, 58]). This makes sense not just for one X, but for the whole category $\mathbf{Sm}/k$. Once we have selected a Grothendieck topology τ on $\mathbf{Sm}/k$, we have the notion of a *sheaf* for τ, namely a presheaf P such that, for all covering families $\{f_\alpha : U_\alpha \to U\}$, the sequence

$$0 \to P(U) \to \prod_{\alpha} P(U_\alpha) \to \prod_{\alpha,\beta} P(U_\alpha \times_U U_\beta)$$

is exact, where the first map is the product of the "restriction" maps for $U_\alpha \to U$, and the second is the difference of the two restriction maps for $p_1 : U_\alpha \times_U U_\beta \to U_\alpha$, $p_2 : U_\alpha \times_U U_\beta \to U_\beta$.

The topology we need is the *Nisnevich topology.* In this topology, a covering family of U is a collection of étale (i.e., flat and unramified) maps $\{f_\alpha : U_\alpha \to U\}$ such that, for each finitely generated field extension L of k, the map on L-points

$$\amalg_\alpha U_\alpha(L) \to U(L)$$

is surjective. Another way to say the same thing: for each point $x \in U$, there is an α and a point $x_\alpha \in U_\alpha$ with $f_\alpha(x_\alpha) = x$ and with $f_\alpha^* : k(x) \to k(x_\alpha)$ an isomorphism on the residue fields.

Example 3.2.4. Let $U = \mathbb{A}^1 \setminus \{0\} = \operatorname{Spec} k[t, t^{-1}]$. Let

$$V = \operatorname{Spec} k[t, t^{-1}, y]/y^2 - t,$$

with projection $f : V \to U$ dual to the inclusion of rings of functions $k[t, t^{-1}] \to k[t, t^{-1}, y]/y^2 - t$. Since $d(y^2 - t) = 2ydy - dt$ (and we have inverted t, so $y = 0$ is not in V), we see that $V \to U$ is étale and surjective. However, not every residue field extension is an isomorphism, for example, the extension over the generic point $\operatorname{Spec} k(t)$ is the degree two extension $k(t)[y]/y^2 - t$. Thus $V \to U$ is *not* a Nisnevich cover. However, if we add the open subscheme $V' = \operatorname{Spec} k[t, t^{-1}, (t-1)^{-1}] = \mathbb{A}^1 \setminus \{0, 1\}$, then $\{V \to U, V' \to U\}$ is a Nisnevich cover. In fact, $V' \to V$ covers all points $t \neq 1$, and over $t = 1$, we can solve the equation $y^2 - t$ (namely $y = \pm 1$) so the residue field extension

$$k = k(1) \subset k(1, 1) = k$$

is an isomorphism.

The category $Sh^{tr}_{\mathrm{Nis}}(k)$ is the subcategory of $\mathrm{PST}(k)$ consisting of the presheaves P whose restriction to $\mathbf{Sm}/k$ is a Nisnevich sheaf. Most importantly for us, the presheaves $X \mapsto C_n(Y)(X)$ are Nisnevich sheaves, so $C^{\mathrm{Sus}}_*(Y)$ forms a complex in the abelian category $Sh^{tr}_{\mathrm{Nis}}(k)$. Thus, we can consider the complexes $C^{\mathrm{Sus}}_*(\mathbb{P}^n/\mathbb{P}^{n-1})$ as substitutes for the singular cochain complex functor $X \mapsto C^*_{\mathrm{sing}}(X)$.

In fact, one can further modify the category of complexes of sheaves with transfer, just as we modify the category of complexes of abelian groups, by inverting certain maps. We first invert the maps which are *Nisnevich local quasi-isomorphisms*, i.e., maps $f : C \to C'$ which are quasi-isomorphisms on all Nisnevich stalks. We also need to invert maps to impose the $\mathbb{A}^1$-*homotopy invariance property.* To do this, the projection map $X \times \mathbb{A}^1 \to X$ gives a map of sheaves

$$p_1^* : C_0^{\mathrm{Sus}}(X \times \mathbb{A}^1) \to C_0^{\mathrm{Sus}}(X)$$

and we invert all these maps. This gives us Voevodsky's category of motives $DM_-^{\mathrm{eff}}(k)$ [57, Chapter 5, section 3] (the subscript $_-$ comes from a technical point which we have not mentioned, namely, that we only consider complexes of sheaves C^* with $C^n = 0$ for $n >> 0$).

We have the analog of the singular chain complex functor by sending X to the image of complex $C_*^{\mathrm{Sus}}(X)$ in $DM_-^{\mathrm{eff}}(k)$; we write this as $M(X)$. We denote the complex $C_*^{\mathrm{Sus}}(\mathbb{P}^n/\mathbb{P}^{n-1})[-2n]$ by $\mathbb{Z}(n)$, and one has the following formula for motivic cohomology:

$$H^p(X, \mathbb{Z}(q)) \cong \mathrm{Hom}_{DM_-^{\mathrm{eff}}(k)}(M(X), \mathbb{Z}(q)[p]),$$

analogous to the formula for cohomology of a topological space T with coefficients in an abelian group A:

$$H^p(T, A) \cong \mathrm{Hom}_{D^-(\mathbf{Ab})}(C_*(T), A[p]).$$

Here the operation $C \mapsto C[p]$ is the usual shift operator on complexes:

$$C[p]^n := C^{n+p}.$$

It is a nice exercise to show that $\mathrm{Hom}_{D(\mathbf{Ab})}(C_*, A[p])$ calculates the pth cohomology of the Hom-complex $\mathrm{Hom}_{\mathbf{Ab}}(C_*, A)$ if C_* is a complex of free abelian groups.

Remark 3.2.5. Voevodsky's definition of $DM_-^{\mathrm{eff}}(k)$ is different from what we have presented here. His definition starts with the derived category $D^-(Sh_{\mathrm{Nis}}^{tr}(k))$, which is formed from the category of complexes of Nisnevich sheaves with transfer by inverting Nisnevich local quasi-isomorphisms, just as we did above. Now, instead of inverting the $\mathbb{A}^1$-weak equivalences, Voevodsky defines $DM_-^{\mathrm{eff}}(k)$ to be the full subcategory of $D^-(Sh_{\mathrm{Nis}}^{tr}(k))$ consisting of those complexes C^* whose Nisnevich cohomology sheaves $\mathcal{H}^n(C^*)$ are homotopy invariant:

$$\mathcal{H}^n(C^*)(X) \cong \mathcal{H}^n(C^*)(X \times \mathbb{A}^1)$$

for all $X \in \mathbf{Sm}/k$. A fundamental theorem [57, Chapter 5, Theorem 3.1.12] of Voevodsky's implies that this condition does in fact define a triangulated subcategory of $D^-(Sh^{tr}_{\mathrm{Nis}}(k))$. Voevodsky goes on to prove that the definition of $DM^{\mathrm{eff}}_-(k)$ we gave here, as a localization of $D^-(Sh^{tr}_{\mathrm{Nis}}(k))$, is equivalent to his definition as a subcategory of $D^-(Sh^{tr}_{\mathrm{Nis}}(k))$ (see [57, Chapter 5, Theorem 3.2.6]).

Beside the book [57], we recommend [27] as a good source for further reading on Voevodsky's category of motives and [10] for a nice overview of related topics.

4. Homotopy theory

We turn to a discussion of classical homotopy theory and the parallel world of motivic homotopy theory.

4.1. Topology

In our bird's eye view of homotopy theory, the basic ingredients are:

1. Spaces **Spc** (i.e., simplicial sets).

2. The (unstable) homotopy category $\mathcal{H}$. This construction relies on two special spaces: the interval $I = [0,1]$ and the circle $S^1 := I/\{0,1\}$. Using I, one defines the notion of homotopy of maps: $f, g : X \to Y$ are homotopic if there is a map $H : X \times I \to Y$ with $f = H \circ i_0$, $g = H \circ i_1$. The homotopy relation $\sim$ leads in turn to the notion of *homotopy equivalence of spaces*, $f : X \to Y$ being a homotopy equivalence if there is a homotopy inverse $g : Y \to X$, i.e., $gf \sim \mathrm{id}_X$, $fg \sim \mathrm{id}_Y$.

The circle S^1 plus the homotopy relation leads to the *homotopy groups* of a pointed space, as follows: S^1 has as natural base-point $*$ the image of $\{0,1\}$, For pointed spaces $(X,*)$, $(Y,*)$, we have the smash product

$$X \wedge Y := X \times Y / X \times * \cup * \times Y.$$

The n-sphere S^n is just the n-fold smash product of S^1, and the nth homotopy group $\pi_n(X,*)$ is the set of homotopy classes of pointed maps $(S^n,*) \to (X,*)$. This works even for $n = 0$, where $S^0 := \{0,1\}$ with base-point 0.

Once we have homotopy groups, we can define the *unstable homotopy category* $\mathcal{H}$ by inverting the weak equivalences in $\mathbf{Spc}(k)$:

$$\mathcal{H} := \mathbf{Spc}[WE^{-1}]$$

where WE is the collection of *weak equivalences*: a map of spaces $f : X \to Y$ which induces an isomorphism on all homotopy groups. Since we are dealing with simplicial sets, it turns out the WE is just the collection of all homotopy equivalences, but never mind. Performing the same construction on the category of pointed spaces gives us the pointed homotopy category $\mathcal{H}_\bullet$. We write

$$[A, B] := \mathrm{Hom}_{\mathcal{H}_\bullet}(A, B).$$

3. Generalized cohomology and the stable homotopy category $\mathcal{SH}$. To represent cohomological functors E^* from $\mathbf{Spc}$ to graded abelian groups, one enlarges $\mathbf{Spc}$ to *spectra* $\mathbf{Spt}$ and $\mathcal{H}$ to the *stable homotopy category* $\mathcal{SH}$. Roughly, we want functors which satisfy

$$E^n(X) = E^{n+1}(\Sigma X)$$

where $\Sigma X := X \wedge S^1$, so we replace a space E with a sequence of pointed spaces $E_0, E_1, \ldots$ together with bonding maps $\epsilon_n : \Sigma E_n \to E_{n+1}$.

Given a pointed space X, we use the bonding maps to send $[\Sigma^{N-n} X, E_N]$ to $[\Sigma^{N-n+1} X, E_{N+1}]$, giving us the inductive system of sets

$$N \mapsto [\Sigma^{N-n} X, E_N].$$

We define $E^n(X)$ by

$$E^n(X) := \lim_{N \to \infty} [\Sigma^{N-n} X, E_N];$$

it is easy to see that $X \mapsto E^n(X)$ satisfies the suspension property we want. Such a gadget $E := (E_0, E_1, \ldots)$, together with the maps $\epsilon_n : \Sigma E_n \to E_{n+1}$, is a *spectrum*. This gives us the category of spectra, $\mathbf{Spt}$, by taking a morphism $E \to F$ to be a sequences of maps $E_n \to F_n$ which respect the bonding maps.

Since S^1 is a co-group and S^2 a commutative co-group (in $\mathcal{H}_\bullet$), the $E^n(X)$ are not just sets, but have a natural abelian group structure. Note also that $E^n(X)$ is defined for *all* $n \in \mathbb{Z}$, not just $n \geq 0$. Taking $X = S^0$, we have the *stable homotopy groups* of E:

$$\pi_n^s(E) := E^{-n}(S^0).$$

A map of spectra $E \to F$ is a *stable weak equivalence* if $\pi_n^s(E) \to \pi_n^s(F)$ is an isomorphism for all n; inverting the stable weak equivalences in **Spt** gives us the *stable homotopy category* $\mathcal{SH} := \mathbf{Spt}[sWE^{-1}]$.

The Brown representability theorem (see, e.g., [43, Chapter 7, section 7]) tells us that $\mathcal{SH}$ is exactly the category of (generalized) cohomology theories: given a cohomology theory E^* on spectra, there is an $\mathcal{E} \in \mathcal{SH}$ (unique up to unique isomorphism) with $E^*(F) = \mathrm{Hom}_{\mathcal{SH}}(F, \Sigma^*\mathcal{E})$ for all spectra F. Some well-known cohomology theories and their representing objects are:

1. Singular cohomology $H^*(-, A)$ is represented by the Eilenberg-MacLane spectrum $\mathcal{H}A$, with $\mathcal{H}A_n = K(n, A)$, the space with
$$\pi_m(K(n,A)) = \begin{cases} A & \text{for } n = m \\ 0 & \text{else.} \end{cases}$$
For $A = \mathbb{Z}$, we can use the Dold-Thom theorem to give a nice model for $K(n, \mathbb{Z})$: $K(n, \mathbb{Z}) = \mathrm{Sym}^\infty S^n$.
2. Topological K-theory. A complex vector bundle on a space X is given by a map into the space with a universal bundle, namely BU, the (doubly) infinite Grassmann variety. Bott periodicity says $\Omega^2 BU \cong BU$, which gives a map $\Sigma^2 BU \to BU$. Thus, topological K-theory is represented by the *K-theory spectrum*
$$K := (BU \times \mathbb{Z}, \Sigma BU, BU, \Sigma BU, \ldots)$$
using the Bott map $\Sigma^2 BU \to BU$ for ϵ_{2n+1} and the identity for ϵ_{2n}.
3. Complex cobordism MU^* is represented by the *Thom spectrum* MU, formed from the Thom spaces of the universal bundles $E_n \to BU_n$:
$$MU := (pt, S^1, MU_1, \Sigma MU_1, MU_2, \Sigma MU_2, \ldots)$$
where MU_n is the Thom space
$$MU_n := Th(E_n) = E_n/(E_n \setminus 0_{BU_n}).$$
We need a map $\Sigma^2 MU_n \to MU_{n+1}$; this comes from the embedding $i_n : BU_n \to BU_{n+1}$, noting that $i_n^* E_{n+1} = E_n \oplus \mathbb{C}$, so
$$Th(i_n^* E_{n+1}) = Th(E_n) \wedge S^2.$$
This gives us the map
$$\Sigma^2 Th(E_n) = Th(E_n) \wedge S^2 = Th(i_n^* E_{n+1}) \xrightarrow{i_{n*}} Th(E_{n+1}).$$

4.2. Algebraic geometry

Morel and Voevodsky [36] have shown us how to translate these basic constructions in homotopy theory into the world of algebraic geometry.

1. Spaces over k, $\mathbf{Spc}(k)$. Taking a hint from our replacement of abelian groups by presheaves on $\mathbf{Sm}/k$, we replace the category of spaces $\mathbf{Spc}$ by the category $\mathbf{Spc}(k)$ of presheaves of spaces on $\mathbf{Sm}/k$, i.e., functors $P : \mathbf{Sm}/k^{\mathrm{op}} \to \mathbf{Spc}$.

The category $\mathbf{Spc}(k)$ mixes the two categories $\mathbf{Spc}$ and $\mathbf{Sm}/k$ together in a very useful way. Sending $X \in \mathbf{Sm}/k$ to the presheaf of sets $Y \mapsto \mathrm{Hom}_{\mathbf{Sm}/k}(Y, X)$ gives a full embedding $i : \mathbf{Sm}/k \to \mathbf{Spc}(k)$. In other words, each smooth k-scheme is a presheaf of *discrete* spaces. Given a space K, we have the constant presheaf: $K(Y) = K$, giving the embedding $\iota : \mathbf{Spc} \to \mathbf{Spc}(k)$.

Even more, $\mathbf{Spc}(k)$ inherits constructions from $\mathbf{Spc}$ by doing them to the "values" of various presheaves. For instance, if P and Q are pointed presheaves of spaces on $\mathbf{Sm}/k$, we can form the presheaf $P \wedge Q$ by

$$(P \wedge Q)(Y) := P(Y) \wedge Q(Y).$$

This allows us to define, e.g., the suspension operator on pointed spaces over k, $\mathbf{Spc}_*(k)$, by

$$\Sigma P := P \wedge S^1.$$

This makes sense even for $P = i(X)$, $X \in \mathbf{Sm}/k$.

To define the homotopy relation, we could just use the homotopy relation on $\mathbf{Spc}$, promoted up to presheaves in the same way as we did smash product or suspension. However, we want to incorporate two new ingredients that come from $\mathbf{Sm}/k$ and not from $\mathbf{Spc}$: the Nisnevich topology and the contractibility of the affine line. We build these into the construction through the homotopy category $\mathcal{H}(k)$.

2. The unstable homotopy category $\mathcal{H}(k)$. To reflect the Nisnevich topology, we have

Definition 4.2.1. A map of spaces over k $f : P \to Q$, is a *Nisnevich local weak equivalence* if, for each for $x \in X \in \mathbf{Sm}/k$, f induces s a weak equivalence of spaces on each Nisnevich stalk $f_x : P_x \to Q_x$.

Here, the Nisnevich stalk of P at x is the limit

$$P_x := \lim_{\to} P(U),$$

where U runs over Nisnevich neighborhoods of x in X, i.e., étale maps $U \to X$ plus a section $x \to U$ over $x \in X$. Inverting all Nisnevich local weak equivalences in $\mathbf{Spc}(k)$ gives us the first approximation $\mathcal{H}_{\mathrm{Nis}}(k)$ to $\mathcal{H}(k)$.

To make $\mathbb{A}^1$ contractible, we first look at spaces for which X and $X \times \mathbb{A}^1$ look the same: call a space Z $\mathbb{A}^1$-*local* if $\mathrm{Hom}_{\mathcal{H}_{\mathrm{Nis}}}(X, Z) \to \mathrm{Hom}_{\mathcal{H}_{\mathrm{Nis}}}(X \times \mathbb{A}^1, Z)$ is an isomorphism for all $X \in \mathbf{Sm}/k$. A map $f : P \to Q$ is then an $\mathbb{A}^1$-*weak equivalence* if $f^* : \mathrm{Hom}_{\mathcal{H}_{\mathrm{Nis}}}(Q, Z) \to \mathrm{Hom}_{\mathcal{H}_{\mathrm{Nis}}}(P, Z)$ is an isomorphism for all $\mathbb{A}^1$-local Z. Let $WE_{\mathbb{A}^1}(k)$ be the $\mathbb{A}^1$-weak equivalences in $\mathbf{Spc}(k)$.

Definition 4.2.2. Define the Morel-Voevodsky unstable homotopy category of spaces over k, $\mathcal{H}(k)$, by

$$\mathcal{H}(k) := \mathcal{H}_{\mathrm{Nis}}(k)[WE_{\mathbb{A}^1}(k)^{-1}],$$

i.e., by inverting all $\mathbb{A}^1$-weak equivalences in $\mathcal{H}_{\mathrm{Nis}}(k)$.

Note that $X \times \mathbb{A}^1 \to X$ is automatically an $\mathbb{A}^1$-weak equivalence, so we have made $\mathbb{A}^1$ "contractible" by this process. We have pointed versions, $\mathbf{Spc}_\bullet(k)$, $\mathcal{H}_{\mathrm{Nis}\bullet}(k)$ and $\mathcal{H}_\bullet(k)$, defined similarly.

By the way, what we've done is a standard procedure in the theory of model categories, called *Bousfield localization* (see e.g. [14]). Under appropriate hypotheses, this gives a good way of selecting *all* the maps one needs to invert, given a collection of maps that one wants to invert: in our case, we wanted to invert the maps $X \times \mathbb{A}^1 \to X$, and Bousfield tells us we should really invert all the $\mathbb{A}^1$-weak equivalences.

One basic result, the purity theory of Morel-Voevodsky [36, Theorem 3.2.23], makes the classical tubular neighborhood construction available in $\mathcal{H}_\bullet(k)$. As we have mentioned before, for $i : V \to W$ an embedding of manifolds, a tubular neighborhood T of V in W is homeomorphic to the normal bundle N of V in W, leading to the homeomorphism

$$W/(W \setminus V) \cong N/(N \setminus 0_V) =: Th(N)$$

where 0_V is the zero-section of N. The analog of this is not true in $\mathbf{Spc}_\bullet(k)$, but is so in $\mathcal{H}_\bullet(k)$: for $i : Y \to X$ a closed embedding of algebraic manifolds, with normal bundle N_i, we have an isomorphism in $\mathcal{H}_\bullet(k)$

$$X/(X \setminus Y) \cong N_i/(N_i \setminus 0_Y) =: Th(N_i).$$

If now N_i is a trivial bundle, then, as we saw before, we have an isomorphism in $\mathcal{H}_\bullet(k)$

$$Th(N_i) \cong \mathbb{P}^d/P^{d-1} \wedge Y_+$$

with $d = \mathrm{codim}_X Y$, replacing the homeomorphism $Th(N) \cong S^n \wedge V_+$, $n = \mathrm{codim}_W V$ in the topological setting (assuming N is the trivial bundle).

3. The stable homotopy category of T-spectra. Replacing $\mathbf{Spc}_\bullet$ with $\mathbf{Spc}_\bullet(k)$ and $S^1 = Th(\mathbb{R})$ with $\mathbb{P}^1 = Th(\mathbb{A}^1)$ in our definition of spectra gives us the category of *T-spectra over k*, $\mathbf{Spt}(k)$. Objects are sequences of pointed spaces over k, $E_0, E_1, \ldots$, with bonding maps

$$\epsilon_n : E_n \wedge \mathbb{P}^1 \to E_{n+1}.$$

Morphisms $E \to F$ are as before families of maps $E_n \to F_n$ commuting with the bonding maps. Noting that $\mathbb{P}^1$ is an $S^{2,1}$, we have the presheaf of bi-graded stable homotopy groups

$$\pi^s_{a,b}(E)(X) := \varinjlim_N \mathrm{Hom}_{\mathcal{H}_\bullet(k)}(S^{a+2N,b+N} \wedge X_+, E_N);\ X \in \mathbf{Sm}/k,$$

and the associated Nisnevich sheaf $\pi^s_{a,b}(E)_{\mathrm{Nis}}$.

A *stable weak equivalence*is a map $f : E \to F$ of T-spectra which induces an isomorphism of sheaves $\pi^s_{a,b}(E)_{\mathrm{Nis}} \to \pi^s_{a,b}(F)_{\mathrm{Nis}}$ for all a, b. This gives us the stable homotopy category of T-spectra, $\mathcal{SH}(k)$, by

$$\mathcal{SH}(k) := \mathbf{Spt}(k)[sWE^{-1}].$$

The operator $- \wedge \mathbb{P}^1$ on $\mathcal{H}_\bullet(k)$ extends to the invertible "T-suspension" operator Σ_T on $\mathcal{SH}(k)$.

Thus, the objects E of $\mathcal{SH}(k)$ represent *bi-graded* cohomology theories on $\mathbf{Sm}/k$:

$$E^{a,b}(X) := \varinjlim_N \mathrm{Hom}_{\mathcal{H}_\bullet(k)}(S^{2N-a,N-b} \wedge X, E_N) = \pi^s_{-a,-b}(E)(X).$$

All the theories we have described in the topological setting have algebraic analogs:

1. Motivic cohomology. Replacing the space $K(n, \mathbb{Z}) = \mathrm{Sym}^\infty S^n$ with the space over k, $\mathrm{Sym}^\infty(\mathbb{P}^n/\mathbb{P}^{n-1})$, gives us the *motivic cohomology spectrum $\mathcal{H}\mathbb{Z}$*

   $$\mathcal{H}\mathbb{Z} := (\mathrm{Sym}^\infty(S^{0,0}), \mathrm{Sym}^\infty(S^{2,1}), \ldots, \mathrm{Sym}^\infty(S^{2n,n}), \ldots).$$

   At least for k a field of characteristic zero, $\mathcal{H}\mathbb{Z}$ does indeed represent the motivic cohomology defined in the section on cohomology; see the comments on the Eilenberg-MacLane functor below.

2. Algebraic K-theory. Morel-Voevodsky show that the (doubly) infinite Grassmann variety BGL represents the (reduced) algebraic K_0 functor of Grothendieck (see [36, Theorem 4.3.13]). Also, one has the *projective bundle formula* for K_0, which, for $\mathbb{P}^1$, tells you

$$K_0(X \times \mathbb{P}^1) = K_0(X) \oplus K_0(X)$$

or $K_0(X \wedge \mathbb{P}^1) = K_0(X)$. Since $\mathbb{P}^1 = S^{2,1}$, this is just the algebraic form of Bott periodicity. Thus, one can define a T-spectrum

$$\mathcal{K} := (BGL \times \mathbb{Z}, BGL, \ldots)$$

with map $BGL \wedge \mathbb{P}^1 \to BGL$ coming from Bott periodicity. In fact, $\mathcal{K}$ does represent higher algebraic K-theory on $\mathbf{Sm}/k$.

3. Algebraic cobordism MGL. Replacing BU_n with $BGL_n = \mathrm{Gr}(n, \infty)$, we have the purely algebraic universal bundle $E_n \to BGL_n$ and the *Thom spectrum*

$$MGL := (Th(E_0), Th(E_1), \ldots, Th(E_n), \ldots).$$

The associated cohomology theory MGL^{**} is called *algebraic cobordism*. With Morel, we have defined a geometric version Ω^* of this theory (also called algebraic cobordism, see [23]). Ω^* comes with natural maps $\Omega^n(X) \to MGL^{2n,n}(X)$ for $X \in \mathbf{Sm}/k$, which we conjecture are isomorphisms. This should be the analog of the isomorphisms

$$\mathrm{CH}^n(X) \cong H^{2n,n}(X, \mathbb{Z}).$$

Additionally, there is an *Eilenberg-MacLane* functor

$$DM(k) \xrightarrow{EM} \mathcal{SH}(k).$$

The category $DM(k)$ is a "T-spectrum" version of the category of motives $DM_-^{\mathrm{eff}}(k)$ defined in the previous section. The functor EM is the analog of the Eilenberg-MacLane functor $D(\mathbf{Ab}) \to \mathcal{SH}$ in classical homotopy theory. Work of Röndigs-Østvær [40] show that the functor EM identifies $DM(k)$ with the homotopy category of "$\mathcal{H}\mathbb{Z}$-modules"; this shows in particular that motivic cohomology is indeed represented by the T-spectrum $\mathcal{H}\mathbb{Z}$.

5. Applications and perspectives

We conclude our overview of motivic homotopy theory with some of the most striking applications, together with a few thoughts on open problems.

5.1. The Bloch-Kato conjecture

This far-reaching conjecture is concerned with the *Galois symbol*

$$\theta_{n,F,q} : K_q^M(F)/n \to H^q_{\mathrm{Gal}}(F, \mu_n^{\otimes q}).$$

To explain: F is a field, n is an integer prime to the characteristic of F, μ_n is the $\mathrm{Gal}(F)$-module of nth roots of unity in $\bar{F}$, $q \geq 1$ is an integer, $K_q^M(F)$ is the qth Milnor K-group of F, and $H^q_{\mathrm{Gal}}(F, -)$ is Galois cohomology, i.e., group cohomology for the profinite group $\mathrm{Gal}(F)$. The map $\theta_{n,F,q}$ is defined as follows. For $q = 1$, we consider the *Kummer sequence* of $\mathrm{Gal}(F)$-modules

$$1 \to \mu_n \to \bar{F}^\times \xrightarrow{x^n} \bar{F}^\times \to 1.$$

The relevant part of the long exact Galois cohomology sequence is

$$H^0_{\mathrm{Gal}}(F, \bar{F}^\times) \xrightarrow{x^n} H^0_{\mathrm{Gal}}(F, \bar{F}^\times) \xrightarrow{\partial} H^1_{\mathrm{Gal}}(F, \mu_n) \to H^1_{\mathrm{Gal}}(F, \bar{F}^\times).$$

But $H^0_{\mathrm{Gal}}(F, \bar{F}^\times) = F^\times$ and Hilbert's theorem 90 tells us $H^1_{\mathrm{Gal}}(F, \bar{F}^\times) = 0$, so we have the isomorphism

$$F^\times/(F^\times)^n \xrightarrow{\partial} H^1_{\mathrm{Gal}}(F, \mu_n).$$

Now, the Milnor K-groups $K_*^M(F)$ are defined as the quotient of the tensor algebra on the group of units $F^\times$ by the two-sided ideal generated by the elements $x \otimes (1 - x)$, $x \in F \setminus \{0, 1\}$ (the *Steinberg relation*). Since $K_1(F) = F^\times$, we define $\theta_{n,F,1} := \partial$.

In general, we use the cup product in Galois cohomology to define

$$(K_1^M(F)/n)^{\otimes q} \xrightarrow{\theta_{n,F,1}^{\otimes q}} H^1_{\mathrm{Gal}}(F, \mu_n)^{\otimes q} \xrightarrow{\cup} H^q_{\mathrm{Gal}}(F, \mu_n^{\otimes q}).$$

One then shows that this map kills the Steinberg relation, and thus descends to the desired map $\theta_{n,F,q}$.

Milnor [30] noted a connection between quadratic forms (via the Witt group $W(F)$, the group of quadratic forms over k modulo hyperbolic forms), mod 2 Milnor K-theory and Galois cohomology. He conjectured that, not only is $\theta_{2^\ell,F,q}$ an isomorphism for all ℓ, but that another map

$$\psi_{F,q} : K_q^M(F)/2 \to I^q/I^{q+1}$$

is an isomorphism for all F and q. Here $I \subset W(F)$ is the augmentation ideal, i.e., the quadric forms of even rank, and $\psi_{F,q}$ sends the symbol $\{a_1, \ldots, a_q\}$ (the image in K_q^M of the tensor $a_1 \otimes \ldots \otimes a_q$) to the class of the *Pfister quadric form* $\langle\langle a_1, \ldots, a_q\rangle\rangle$. This quadratic form is defined as follows: $\langle\langle a\rangle\rangle$ is the form $x^2 - ay^2$, and $\langle\langle a_1, \ldots, a_q\rangle\rangle$ is the tensor product $\langle\langle a_1\rangle\rangle \otimes \ldots \otimes \langle\langle a_q\rangle\rangle$.

The maps $\theta_{n,F,1}$ are isomorphisms by construction (the Kummer sequence). Tate [47] considered the map

$$\theta_{n,F,2} : K_2^M(F)/n \to H^2_{\mathrm{Gal}}(F, \mu_n^{\otimes 2}),$$

for F a number field, and showed that $\theta_{n,F,2}$ is an isomorphism in this case. Using this as an inductive starting point, Merkurjev and Suslin, in a ground-breaking paper [28], used the K-theory of Severi-Brauer varieties to extend Tate's result to arbitrary fields (but still for $q = 2$).

In their paper on p-adic étale cohomology, Bloch and Kato defined an analog of the Galois symbol for the mod p^n Milnor K-groups in characteristic p, and showed that this analog is an isomorphism. They conjectured that $\theta_{n,F,q}$ is an isomorphism for all n, F, q, giving rise to the *Bloch-Kato conjecture*, subsuming at least part of the Milnor conjecture.

There was little progress on the Bloch-Kato conjecture beyond the Merkurjev-Suslin theorem for quite some time. However, it did slowly become apparent that the Bloch-Kato conjecture was closely related to two other conjectures: the Quillen-Lichtenbaum conjecture, and the Beilinson-Lichtenbaum conjecture.

The Quillen-Lichtenbaum conjecture concerns the map from algebraic K-theory to étale K-theory (with mod n coefficients)

$$\tau_{X,n,q} : K_q(X; \mathbb{Z}/n) \to K_q^{\text{ét}}(X; \mathbb{Z}/n).$$

The conjecture asserts that the comparison map τ should be an isomorphism for $q \geq cd_{n,X} - 1$ and an injection for $q = cd_{n,X} - 2$. Here $cd_{n,X}$ is the étale cohomological dimension of the category of n-torsion sheaves on X. Part of the problem in understanding this conjecture was that the failure of τ to be an isomorphism in low degree made the usual topological approaches difficult. The Beilinson-Lichtenbaum conjectures, in their original form, tried to explain the low-degree problem with τ by positing a theory of motivic cohomology that would have a much better relation with étale cohomology, and whose relation with algebraic K-theory would be given by a spectral sequence, akin to the classical Atiyah-Hirzebruch spectral sequence relating singular cohomology and topological K-theory.

What in hindsight is quite remarkable, is that the present theory of motivic cohomology completely fulfills the conjecture of Beilinson-Lichtenbaum (with one important missing ingredient having to do with another conjecture that is still open, the *Beilinson-Soulé conjecture*); in any case, all the parts of the Beilinson-Lichtenbaum conjecture that are important for their application to the Quillen-Lichtenbaum conjecture are now known.

So, let me rewrite history a bit and recount the facts that we now know about motivic cohomology and algebraic K-theory, to get down to the main fact one needs to settle the Quillen-Lichtenbaum conjecture.

First of all, there is a well-defined comparison map

$$\alpha_{X,n,p,q} : H^p(X, \mathbb{Z}/n(q)) \to H^p_{\text{ét}}(X, \mathbb{Z}/n(q)^{\text{ét}}).$$

Here X is some smooth variety and $H^p(X, \mathbb{Z}/n(q))$ is motivic cohomology with mod n coefficients, defined using Voevodsky's category of motives $DM_-^{\text{eff}}(k)$. The group $H^p_{\text{ét}}(X, \mathbb{Z}/n(q)^{\text{ét}})$ is the étale cohomology of X with coefficients the sheaf $\mathbb{Z}/n(q)^{\text{ét}} := \mu_n^{\otimes q}$. The map $\alpha_{X,n,p,q}$ can be viewed as arising from a change of topologies construction, going from the Nisnevich topology to the étale topology.

For $X = \operatorname{Spec} F$, $H^p_{\text{ét}}(F, \mathbb{Z}/n(q)^{\text{ét}}) = H^p_{\text{Gal}}(F, \mu_n^{\otimes q})$. Also, there is a natural isomorphism

$$H^q(F, \mathbb{Z}(q)) \cong K_q^M(F)$$

(due to Totaro [48] and Nesterenko-Suslin [37]), which leads to an isomorphism $H^q(X, \mathbb{Z}/n(q)) \cong K_q^M(F)/n$. Via these isomorphisms, the change of topology map $\alpha_{X,n,q,q}$ turns out to be the Galois symbol $\theta_{n,F,q}$.

Also, there is a spectral sequence, established in case $X = \operatorname{Spec} F$ by Bloch and Lichtenbaum [6], and extended to arbitrary smooth X by Friedlander-Suslin [11]

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}(-q)) \Longrightarrow K_{-p-q}(X),$$

and a mod n version

$$E_2^{p,q} = H^{p-q}(X, \mathbb{Z}/n(-q)) \Longrightarrow K_{-p-q}(X; \mathbb{Z}/n). \tag{5.1.1}$$

A very similar looking spectral sequence, going from mod n étale cohomology and converging to mod n étale K-theory, was constructed much earlier by Dwyer-Friedlander [12] (there are some tricky convergence questions involved in this latter sequence, but let's ignore them).

All these properties were part of the original conjectures of Beilinson and Lichtenbaum. The remaining parts of the conjecture, relevant for us, were

(1) For $p \leq q$, the comparison map $\alpha_{X,n,p,q} : H^p(X, \mathbb{Z}/n(q)) \to H^p_{\text{ét}}(X, \mathbb{Z}/n(q)^{\text{ét}})$ is an isomorphism.
(2) If $\mathcal{O}$ is a the local ring of a smooth point on a variety of finite type over k, we have $H^{q+1}_{\text{ét}}(\mathcal{O}, \mathbb{Z}(q)^{\text{ét}}) = 0$.

With regard to (2), we should note that it is rather easy to show that the motivic cohomology $H^{q+1}(\mathcal{O}, \mathbb{Z}(q))$ vanishes. The complex $\mathbb{Z}(q)^{\text{ét}}$ is defined by taking the étale sheafification of the complex $C_*^{\text{Sus}}(\mathbb{P}^q/\mathbb{P}^{q-1})$ that computes motivic cohomology.

Also, for $q = 1$, the complex $C_*^{\text{Sus}}(\mathbb{P}^1/\mathbb{P}^0)$ is quasi-isomorphic to $\mathbb{G}_m[-1]$, so the statement $H^{q+1}_{\text{ét}}(\mathcal{O}, \mathbb{Z}(q)^{\text{ét}}) = 0$ is just Hilbert's theorem 90 (extended from fields to smooth local rings). For $q = 2$, and $\mathcal{O} = F$ a field, the vanishing of $H^3_{\text{ét}}(\mathcal{O}, \mathbb{Z}(2)^{\text{ét}})$ is related to a theorem of Merkurjev-Suslin [28]: Suppose F contains the ℓth roots of unity, ℓ a prime. Let a be a non-zero element of F, not an ℓth power in F, let $L = F(a^{1/\ell})$ and let σ be a generator of $\text{Gal}(L/F) \cong \mathbb{Z}/\ell$. Then the sequence

$$K_2(L) \xrightarrow{1-\sigma} K_2(L) \xrightarrow{Nm_{L/F}} K_2(F)$$

is exact. If we replace K_2 with K_1, and note that $K_1(k) = k^\times$ for any field k, the exactness is just the classical statement of Hilbert's theorem 90, so this result is called *Hilbert's theorem 90 for K_2*. In fact, this was the key result used by Merkurjev and Suslin to show that $\theta : K_2(F)/n \to H^2_{\text{Gal}}(F, \mu_n^{\otimes 2})$ is an isomorphism.

Using a comparison of the Atiyah-Hirzebruch type spectral sequence (5.1.1) with its étale analog, it is not hard to show that the Beilinson-Lichtenbaum conjecture implies the Quillen-Lichtenbaum conjecture; in short, the fact that the comparison between motivic and étale cohomology is an isomorphism only in degrees up to the weight, lead to the "error" in the comparison between algebraic and étale K-theory in low degree.

Finally, we should note that the Beilinson-Lichtenbaum conjecture part (1), for $p = q$ and $X = \text{Spec}\, F$, is just the Bloch-Kato conjecture for q. Thus, the Beilinson-Lichtenbaum conjecture implies the Bloch-Kato conjecture as a special case.

In fact, the converse proved to be true as well: Suslin and Voevodsky showed

Theorem 5.1.1 (Suslin-Voevodsky [46]**).** *Let k be a field of characteristic zero, $m \geq 1$ an integer Suppose that the Bloch-Kato conjectures are true for a fixed value of n, for all fields F finitely generated over k, and for all $q \leq m$. Then the Beilinson-Lichtenbaum conjecture* (1) *is true for n, for all $q \leq m$ and for all $X \in \mathbf{Sm}/k$.*

The characteristic zero hypothesis was removed by Geisser-Levine [13]. Thus, everything boils down to the Bloch-Kato conjecture (in fact, the arguments for this also handle part (2) of the Beilinson-Lichtenbaum conjectures, but we will ignore this point here).

Voevodsky announced a proof of the Milnor conjecture around 1996 (see [52] for the published version). The approach he used was in rough outline the same as that used by Merkurjev and Suslin in their proof for $q = 2$: One understands the Galois symbol fairly well on symbols $\{a_1, \ldots, a_q\}$. To understand the Galois symbol on a general element, written as a sum of symbols, one would like to make a special field extension that kills exactly one symbol (and all its multiples) and then use induction on the number of terms in the sum. In the case $q = 2$, Merkurjev and Suslin used the function field of a Severi-Brauer variety for this purpose; in the case of arbitrary q, but with $n = 2$, the hypersurfaces defined by the Pfister quadratic forms play this role. Voevodsky [52] used results of M. Rost on K-cohomology of Pfister quadrics, and combined this information with a degree- and weight-shifting argument that relied on "motivic Steenrod operations". This settled the Milnor conjecture, and the Bloch-Kato conjecture for all q and n a power of 2.

To handle the odd primes, Voevodsky and Rost needed first of all to find a replacement for the Pfister quadrics. This turned out to be the *norm varieties*. There are a number of additional complications that arise in the case of $p > 2$ that I will not mention; shortly speaking, a more detailed study of the motivic Steenrod operations, requiring an understanding of the motivic homotopy properties of the symmetric powers of the sphere $S^{2d,d}$, was needed to generalize the argument from 2 to an odd prime. The recent preprints of Voevodsky [49, 50], relying on work Rost [41] (see also the write-up of Suslin-Joukhovitski [45]), plus a contribution of Weibel [59], appear to have settled the full Bloch-Kato conjecture.

5.2. Stable homotopy groups of spheres and Witt groups

One of the most basic invariants of classical homotopy theory is the stable homotopy groups of the spheres. In fact, the identity $\pi_0^s(S^0) = \mathbb{Z}$ is closely related to the existence of the degree map for a proper map of oriented manifolds of the same dimension d: Given $f : M \to N$, choose a regular value $y \in N$, and let $x_1, \ldots, x_r$ be the points in $f^{-1}(y)$. Take a small ball B around y so that $f^{-1}(B) = B_1 \amalg \ldots \amalg B_r$ is a disjoint union of balls, one for each x_i. Using the orientations, we have homotopy equivalences $B_i/\partial B_i \cong S^d$, $B/\partial B \cong S^d$; f induces maps $f_i : B_i/\partial B_i \to B/\partial B$, and thus elements $d_i \in [S^d, S^d] \to \pi_0^s(S^0) = \mathbb{Z}$. The sum of the d_i is the degree of f; one shows that this is independent of the choice of y.

J. Lannes (see [34, 32]) noted that one could make a similar construction in algebraic geometry (at least for self-maps of $\mathbb{P}^1$): Let $f : \mathbb{P}^1_k \to \mathbb{P}^1_k$ be a morphism, take a regular value $y \in \mathbb{P}^1(k)$ and let $x_1, \ldots, x_r$ be the points in $f^{-1}(y)$; we assume that y and all the x_i are not ∞, and let t be the standard coordinate on $\mathbb{P}^1 \setminus \{\infty\}$. We have the derivatives $u_i := df/dt_{|x_i} \in k(x_i)^\times$. Define the quadratic form on $k(x_i)$ (considered as a $k(y)$-vector space) by

$$Q_i(a, b) := Tr_{k(x_i)/k(y)}(u_i ab).$$

The orthogonal direct sum of the Q_i defines an element $\deg(f)$ in the Grothendieck-Witt group $GW(k)$ of quadratic forms over $k = k(y)$. One shows that $\deg(f)$ is independent of the choice of y, giving a map from the self-maps of $\mathbb{P}^1_k$ to $GW(k)$. Note that a map $\mathbb{P}^1_k \to \mathbb{P}^1_k$ is a self-map of the weighted sphere $S^{2,1}$, hence gives rise to an element in the stable π_0 of the *motivic sphere spectrum* S_k, i.e., the T-spectrum

$$S_k := (S^0_k, \mathbb{P}^1_k, \mathbb{P}^1_k \wedge \mathbb{P}^1_k, \ldots, (\mathbb{P}^1_k)^{\wedge n}, \ldots).$$

In fact, Morel conjectured, and then proved

Theorem 5.2.1 (Morel [34]**).** *Let k be a field of characteristic $\neq 2$. Then $\pi_0^s(S_k) \cong GW(k)$, where $\pi_0^s(S_k)$ is the group of self-maps of S_k in the motivic stable homotopy category $\mathcal{SH}(k)$.*

I won't say anything about the proof, but let me at least give the map $GW(k) \to [S_k, S_k]$. This is defined as follows: Let $u \in k^\times$ be a non-zero element of k. This gives us the self-map of $\mathbb{P}^1_k$, by sending $(x_0 : x_1)$ to $(x_0 : ux_1)$. Taking the infinite T-suspension of this map gives the element $[u] \in \mathrm{Hom}_{\mathcal{SH}(k)}(S_k, S_k)$. Now, the quadratic forms $\langle u \rangle := ux^2$ generate $GW(k)$, with a well-known set of relations. Morel shows that these relations are satisfied among the $[u]$, giving the map $GW(k) \to \pi_0(S_k)$.

The computation of the higher homotopy groups of the sphere spectrum appears to be completely open at present.

5.3. Algebraic cobordism

In [23], together with Morel, we have defined a geometric theory of *algebraic cobordism*, $X \mapsto \Omega^*(X)$, which is supposed to describe the geometric part of the bi-graded cohomology theory $\mathrm{MGL}^{*,*}$ represented by the motivic Thom spectrum MGL in $\mathcal{SH}(k)$. By "geometric part", I mean the subring

$\oplus_n \mathrm{MGL}^{2n,n}$; the analogous part of motivic cohomology, $\oplus_n H^{2n,n}$, is the classical Chow ring $\oplus_n \mathrm{CH}^n$.

Our definition of Ω^* is motivated by Quillen's work [39] on complex cobordism. There is a geometric description of $MU^n(X)$, for X a C^∞ manifold, by generators and relations, the generators being proper, $\mathbb{C}$-oriented morphisms of manifolds $f : Y \to X$, with $n = \dim X - \dim Y$, and relations that of cobordism: Given a proper $\mathbb{C}$-oriented morphism $F : W \to X \times \mathbb{R}^1$, transverse to $X \times \{0, 1\}$, one identifies the fiber over 0, $F_0 : W_0 \to X$, with the fiber over 1, $F_1 : W_1 \to X$.

The group $\Omega^*(X)$ has a very similar type of generator: $f : Y \to X$, with $Y \in \mathbf{Sm}/k$, and f projective. The relations include a relation similar to that of complex cobordism, with $\mathbb{A}^1$ replacing $\mathbb{R}^1$, but these naive cobordism relations are not sufficient. The original approach with Morel required adding additional generators: $(f : Y \to X; L_1, \ldots, L_r)$, with the L_i line bundles on Y, and f as before. The L_i should be considered as the result of applying *first Chern class operators* to $f : Y \to X$. We impose 3 relations:

1. A dimension axiom: $(f : Y \to X; L_1, \ldots, L_r) = 0$ if $r > \dim_k Y$.
2. A Gysin relation: For $i : D \to X$ the inclusion of a smooth codimension one subvariety, with associated line bundle $L(D)$, we have
$$[\mathrm{id}_X : X \to X; L(D)] = [i : D \to X].$$
3. A formal group law: We extend coefficients to the coefficient ring $\mathbb{L}$ of the universal formal group law, $F_{\mathbb{L}}(u, v) \in \mathbb{L}[[u, v]]$, and formally force the 1st Chern class operator to satisfy the universal formal group law:
$$F_{\mathbb{L}}(c_1(M), c_1(N)) = c_1(M \otimes N).$$

The naive cobordism relation is a special case of (2). We were able to prove a number of nice properties for Ω^*. In particular, Ω^* is the universal theory satisfying our axioms of an *oriented cohomology theory* on $\mathbf{Sm}/k$. As it is easy to see that $\oplus_n \mathrm{MGL}^{2n,n}$ forms an oriented cohomology theory on $\mathbf{Sm}/k$, this gives us a canonical map
$$\theta_X : \Omega^*(X) \to \mathrm{MGL}^{2*,*}(X).$$

Results of Hopkins-Morel show that θ_X is a surjection with torsion kernel, and an isomorphism for $X = \operatorname{Spec} k$. In [21], we have built on the Hopkins-Morel results to show that θ_X is an isomorphism for all smooth, quasi-projective X, so Ω^* does indeed capture the geometric part of MGL-theory.

With Pandharipande [24], we have greatly simplified the presentation of Ω^*. The new generators are the naive ones: $f : Y \to X$. The relations are an extension of the naive cobordism relation: the *double point cobordism* relation. This means that we consider a projective map $f : W \to X \times \mathbb{A}^1$, with $W \in \mathbf{Sm}/k$, and require that $f^{-1}(1)$ is smooth, but that $f^{-1}(0)$ is a union of two smooth components A, B, meeting transversely along their intersection C. Under these conditions, we have the normal bundle $N_{C/A}$ of C in A, giving us the $\mathbb{P}^1$-bundle $\mathbb{P}(f) := \mathbb{P}(\mathcal{O}_C \oplus N_{C/A}) \to C$. The double-point relation is

$$[f^{-1}(1) \to X] \sim [A \to X] + [B \to X] - [\mathbb{P}(f) \to X].$$

Our main result of [24] is that the free abelian group on the isomorphism classes of maps $f : Y \to X$, modulo the double-point cobordism relation, gives a presentation of $\Omega^*(X)$.

As an application, we prove a conjecture stated in [25]: Let X be a smooth projective threefold over $\mathbb{C}$. In case X is a Calabi-Yau threefold, one considers the Hilbert scheme of 0-dimensional closed subschemes of X of degree n, $\mathrm{Hilb}(X, n)$. Using methods of deformation theory, one constructs a virtual fundamental class $[\mathrm{Hilb}(X, n)]^{vir} \in \mathrm{CH}_0(\mathrm{Hilb}(X, n))$; since $\mathrm{Hilb}(X, n)$ is projective, this class has a degree. Form the generating function

$$Z(X, t) := 1 + \sum_{n\geq 1} \deg[\mathrm{Hilb}(X, n)]^{vir} \cdot t^n.$$

The methods of [26] extend the definition of $Z(X, t)$ to a general smooth projective threefold X, by a different method.

One has the purely combinatorial generating function, the *MacMahon function*

$$M(t) := \prod_{n\geq 1} (1 - t^n)^{-n}.$$

This function counts the number of three-dimensional partitions of size n. In [24], we prove

Theorem 5.3.1. *Let X be a smooth projective threefold. Then*

$$Z(X, t) = M(t)^{\deg c_3(T_X \otimes K_X)}.$$

Here T_X is the tangent bundle of X, and K_X is the canonical line bundle. Our proof goes as follows. It is shown in [25, 26] that $Z(X, t)$ respects the double-point cobordism relation: Given a double-point cobordism

$W \to X \times \mathbb{A}^1$, let $Y = f^{-1}(1)$, $A \cup B = f^{-1}(0)$. Then

$$Z(Y,t) = \frac{Z(A,t) \cdot Z(B,t)}{Z(\mathbb{P}(f),t)}.$$

Thus, sending X to $Z(X,t) \in (1 + t\mathbb{Z}[[t]])^\times$ descends to a group homomorphism

$$Z : \Omega^{-3}(\mathbb{C}) \to (1 + t\mathbb{Z}[[t]])^\times.$$

Similarly, sending X to $\deg c_3(T_X \otimes K_X)$ descends to a homomorphism $\Omega^{-3}(\mathbb{C}) \to \mathbb{Z}$, so $X \mapsto M(t)^{\deg c_3(T_X \otimes K_X)}$ defines a homomorphism

$$M^? : \Omega^{-3}(\mathbb{C}) \to (1 + t\mathbb{Z}[[t]])^\times.$$

But we know that the canonical map $\Omega^*(\mathbb{C}) \to MU^{2*}(pt) = \mathbb{L}^*$ is an isomorphism (by one of the main results of [23]), and from topology, we know that $\mathbb{L}^{-3} \otimes \mathbb{Q}$ is the $\mathbb{Q}$-vector space generated by $[\mathbb{P}^3]$, $[\mathbb{P}^2 \times \mathbb{P}^1]$ and $[\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1]$. Furthermore, the identity $Z(X,t) = M(t)^{\deg c_3(T_X \otimes K_X)}$ has been verified in [25] for toric varieties, so the two homomorphisms Z and $M^?$ agree on a $\mathbb{Q}$-basis, and hence are identical.

We hope that this new presentation of Ω^* can be extended to give a reasonably understandable presentation of all of $\mathrm{MGL}^{*,*}$.

5.4. The motivic Postnikov tower

For a spectrum E, we have the $n-1$-connected covers

$$p_n : E\langle n\rangle \to E,$$

characterized by the fact that p_n is an isomorphism on homotopy groups π^s_m for $m \geq n$, and that $\pi^s_m E\langle n\rangle = 0$ for $m < n$. By fitting the $n-1$-connected covers $E\langle n\rangle \to E$ of a spectrum E together, one builds the classical *Postnikov tower* in the stable homotopy category $\mathcal{SH}$

$$\ldots \to E\langle n+1\rangle \to E\langle n\rangle \to \ldots \to E.$$

The nth layer in this tower (i.e., the homotopy cofiber of $E\langle n+1\rangle \to E\langle n\rangle$) is just the shifted Eilenberg-MacLane spectrum $\Sigma^n EM(\pi^s_n E)$. Thus, the Postnikov tower shows how to break apart E into the elementary pieces $\Sigma^n EM(\pi^s_n E)$.

One can give a more categorical description of the Postnikov tower, by consider the full subcategories $\Sigma^n\mathcal{SH}^{\mathrm{eff}}$ of $\mathcal{SH}$ consisting of the $n-1$-connected spectra. The inclusion $i_n : \Sigma^n\mathcal{SH}^{\mathrm{eff}} \to \mathcal{SH}$ has a right adjoint $r_n : \mathcal{SH} \to \Sigma^n\mathcal{SH}^{\mathrm{eff}}$, and

$$E\langle n\rangle = i_n r_n E,$$

with map $E\langle n\rangle \to E$ given by the co-unit of the adjunction.

Noting this, Voevodsky defined a *motivic Postnikov tower* in $\mathcal{SH}(k)$, by replacing $\mathcal{SH}^{\mathrm{eff}}$ with the localizing subcategory $\mathcal{SH}^{\mathrm{eff}}(k)$ of $\mathcal{SH}(k)$ generated by the T-suspension spectra of smooth k-schemes, and letting $\Sigma^n_T\mathcal{SH}^{\mathrm{eff}}(k)$ similarly be the localizing subcategory of $\mathcal{SH}(k)$ generated by the nth T-suspension of objects in $\mathcal{SH}^{\mathrm{eff}}(k)$. Just as in the classical case, the inclusion $i_n : \Sigma^n_T\mathcal{SH}^{\mathrm{eff}}(k) \to \mathcal{SH}(k)$ admits a right adjoint $r_n : \mathcal{SH}(k) \to \Sigma^n_T\mathcal{SH}^{\mathrm{eff}}(k)$, giving the truncation functor

$$f_n : \mathcal{SH}(k) \to \ \mathcal{SH}(k),$$

$f_n := i_n r_n$. Setting $\mathcal{E}\langle n\rangle := f_n\mathcal{E}$ for $\mathcal{E} \in \mathcal{SH}(k)$, we have the *motivic Postnikov tower* of $\mathcal{E}$:

$$\ldots \to \mathcal{E}\langle n+1\rangle \to \mathcal{E}\langle n\rangle \to \ldots \to \mathcal{E}.$$

What about the layers

$$s_n\mathcal{E} := \mathrm{cofib}[\mathcal{E}\langle n+1\rangle \to \mathcal{E}\langle n\rangle]?$$

Results of Voevodsky [51], Röndigs-Østvær [40] and Pelaez-Menaldo [38] imply that $s_n\mathcal{E} \cong \Sigma^n_T EM(\pi^\mu_n\mathcal{E})$, for a well-defined motive $\pi^\mu_n\mathcal{E}$, the nth *homotopy motive* of $\mathcal{E}$. Thus, the abelian group $\pi^s_n E \in D(\mathbf{Ab})$ gets replaced with the motive $\pi^\mu_n\mathcal{E} \in DM(k)$.

In fact, just as an abelian group is a special object in the derived category $D(\mathbf{Ab})$, the homotopy motive $\pi^\mu_n\mathcal{E}$ is a special type of motive, a *birational motive*. These motives, studied by Huber, Kahn and Sujatha [15, 19] are the motives M characterized by the property that the restriction map

$$\mathrm{Hom}_{DM(k)}(M(X), M[n]) \to \mathrm{Hom}_{DM(k)}(M(U), M[n])$$

is an isomorphism for all open dense immersions $U \to X$ in $\mathbf{Sm}/k$, and for all n. Thus, the birational motives are in some sense locally constant in the Zariski topology. The most evident example of such is the constant sheaf with transfers $\mathbb{Z}$, but there are many other more exotic examples.

If we evaluate the motivic Postnikov tower (or rather, its associated tower of 0-spectra) on some $X \in \mathbf{Sm}/k$, we get the tower of spectra

$$\ldots \to \mathcal{E}\langle n+1\rangle^0(X) \to \mathcal{E}\langle n\rangle^0(X) \to \ldots \to \mathcal{E}^0(X).$$

with layers the complexes $\pi^\mu_n(X)(n)[2n](X)$. The resulting spectral sequence is the *motivic Atiyah-Hirzebruch spectral sequence*

$$E_2^{p,q} := H^{p-q}(X, \pi^\mu_{-q}(n-q)) \Longrightarrow \mathcal{E}^{p+q,n}(X).$$

Perhaps the first T-spectrum for which the homotopy motives was calculated was the T-spectrum $\mathcal{K}$ representing algebraic K-theory. The computation (see e.g. [22]) gives

$$\pi_n^{\mu}\mathcal{K} = \mathbb{Z}$$

for all n. The Atiyah-Hirzebruch spectral sequence (with $n = 0$) is

$$E_2^{p,q} := H^{p-q}(X, \mathbb{Z}(-q)) \Longrightarrow K^{p+q,0}(X) = K_{-p-q}(X).$$

This is exactly the spectral sequence of Friedlander-Suslin [11] (generalizing the Bloch-Lichtenbaum spectral sequence [6] for $X = \operatorname{Spec} F$).

Together with B. Kahn [18], we have examined the layers and spectral sequence for the K-theory of a central simple algebra A over K, as well as for the motive of the Severi-Brauer variety $X = SB(A)$ associated to A. We compute the homotopy motives for K^A as

$$\pi_n^{\mu}(K^A) = \mathbb{Z}_A$$

where $\mathbb{Z}_A \subset \mathbb{Z}$ is the subsheaf with value on a field F the ideal in $\mathbb{Z}$ generated by the *index* of $A \otimes_k F$. Recall that, for a central simple algebra A over k, $A \cong M_n(D)$ for some uniquely determined division algebra D over k, and the index e_A is determined by the identity $e_A^2 = \dim_k D$.

We also compute the homotopy motives for the motive of $X = SB(A)$, in case A has prime degree ℓ over k, getting the sheaves $\mathbb{Z}_{A^{\otimes i}}$, $i = 0, \ldots, \ell - 1$. Using this information, we show

Theorem 5.4.1 (Kahn-Levine [18, Theorem 6.2.2]**).** *Let A be a central simple algebra of square-free index over a field k. Then the reduced norm map*

$$K_2(A) \to K_2(k)$$

is injective.

5.5. Perspectives for the future

Recent work of J. Ayoub [5] has put the theory of the motivic stable homotopy category on a very good functorial footing. The motivic stable homotopy category of Morel-Voevodsky is actually defined over a general base-scheme S, giving a functor from schemes (quasi-projective over a fixed base B) to triangulated categories. Ayoub has shown how the functor $S \mapsto \mathcal{SH}(S)$ fulfills Grothendieck's yoga of six operations, as re-formulated by Voevodsky in his notion of *cross functors*. In a nutshell, this means that, in addition to the pull-back functor

$$f^* : \mathcal{SH}(S) \to \mathcal{SH}(T)$$

for each morphism of schemes $f : T \to S$, one has push-forward functors $f_* : \mathcal{SH}(T) \to \mathcal{SH}(S)$, as well as the functors "with compact support"

$$f_! : \mathcal{SH}(T) \to \mathcal{SH}(S); \quad f^! : \mathcal{SH}(S) \to \mathcal{SH}(T),$$

and also internal Hom and tensor product operations, all satisfying the compatibilities that are known for, e.g., the derived category of sheaves on a topological space. A parallel theory is still in the works for motives; work of Cisinski-Déglise [7] goes a long way toward verifying the necessary properties in this setting.

One motivation for pursuing this line is the hope that degeneration techniques could help in showing that geometric motives are all "finite dimensional" in the sense of Kimura [20] and O'Sullivan (see [4]). This property has remarkable consequences, implying for example Bloch's conjecture on the finite dimensionality of 0-cycles for surfaces with $p_g = 0$, and much more.

5.6. The motivic fundamental group

In fact, there are several different theories of the motivic fundamental group. One theory is based on the category of mixed Tate motives. The triangulated category of mixed Tate motives is the full subcategory $\mathrm{DMT}(k)$ of $DM_-^{\mathrm{eff}}(k)_{\mathbb{Q}}$ generated by the Tate objects $\mathbb{Q}(n)$. Assuming the Beilinson-Soulé vanishing conjectures for k (for instance, if k is a number field), $\mathrm{DMT}(k)$ contains an abelian category, $\mathrm{MT}(k)$, generated by the $\mathbb{Q}(n)$ and all extensions. $\mathrm{MT}(k)$ (when it exists) is in fact a $\mathbb{Q}$-Tannakian category, hence is equivalent to the category of representations of its Tannaka group $\mathrm{Gal}(\mathrm{MT}(k))$.

For $X = \mathbb{P}^1_k \setminus S$, S a set of k-points of $\mathbb{P}^1_k$, Deligne and Goncharov [8] have defined a group-scheme object, $\pi_1^{DG}(X, x)$, over $\mathrm{MT}(k)$, the *motivic fundamental group*. Here the base-point x is either a k-point of X or a so-called "tangential base-point". The group-scheme $\pi_1^{DG}(X, x)$ over $\mathrm{MT}(k)$ gives rise to a group-scheme over $\mathbb{Q}$ upon applying the Betti realization of $\mathrm{MT}(k)$; what one gets is the Malčev completion of the topological fundamental group $\pi_1(X(\mathbb{C}), x)$. This gives a motivic version of the category of uni-potent local systems on X. It is at present not clear how to extend this to a larger class of local systems, although work of Katzarkov-Panteev-Toen [17] suggests an approach to a motivic fundamental group that reflects *all* local systems on $X(\mathbb{C})$.

In a rather different direction, Morel [31] has considered the homotopy groups of spaces over k. These are actually Nisnevich sheaves of groups on

$\mathbf{Sm}/k$. For example, for a space X, one has the sheaf $\pi_0^{\mathbb{A}^1}(X)$, this being the sheaf associated to the presheaf

$$U \mapsto \mathrm{Hom}_{\mathcal{H}(k)}(U, X).$$

If X is itself in $\mathbf{Sm}/k$, we have the evident map $\mathrm{Hom}_{\mathbf{Sm}/k}(U, X) \to \pi_0^{\mathbb{A}^1}(X)(U)$, so a k-point of X gives a global section of $\pi_0^{\mathbb{A}^1}(X)$. It does not seem to be known if $X(k) \to \Gamma(\pi_0^{\mathbb{A}^1}(X))$ is always surjective.

If X is smooth and projective, and k is algebraically closed, then one has $\pi_0^{\mathbb{A}^1}(X)(k) = \{*\}$ if X is *rationally connected*, i.e., if each two points of X are connected by a chain of rational curves on X. The converse is apparently not known.

Morel [31] has made computations of the $\pi_1^{\mathbb{A}^1}$ for some varieties. He notes that a $\mathbb{G}_m$-bundle $E \to X$ has the unique homotopy lifting property of a covering space, hence contributes to $\pi_1^{\mathbb{A}^1}(X)$. In particular, no smooth projective variety $X \subset \mathbb{P}^N$ has a trivial $\pi_1^{\mathbb{A}^1}$, since the $\mathbb{G}_m$-bundle associated to $\mathcal{O}_X(1)$ gives a $\mathbb{G}_m$-quotient of $\pi_1^{\mathbb{A}^1}$. Morel [31] shows that $\pi_1^{\mathbb{A}^1}(\mathbb{P}^n) = \mathbb{G}_m$ for $n \geq 2$, but that the surjection $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \to \mathbb{G}_m$ has a non-trivial kernel, given by the Milnor-Witt sheaf $\mathcal{K}_2^{MW}$.

Thus, we have a sharp contrast with the picture given by the topology of varieties over $\mathbb{C}$. It would be interesting to find a condition on say smooth projective hypersurfaces that implies the "triviality" of $\pi_1^{\mathbb{A}^1}$, or more generally, to find a replacement for Morse theory and the classical Lefschetz theorems on hyperplane sections.

References

[1] **Théorie des topos et cohomologie étale des schémas**. Tome 1. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4). Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat. Lecture Notes in Mathematics, **269**. Springer-Verlag, Berlin-New York, 1972.

[2] **Motives**. Proceedings of the AMS-IMS-SIAM Joint Summer Research Conference held at the University of Washington, Seattle, Washington, July 20–August 2, 1991. Edited by Uwe Jannsen, Steven Kleiman and Jean-Pierre Serre. Proceedings of Symposia in Pure Mathematics, **55**, Parts 1 and 2. American Mathematical Society, Providence, RI, 1994.

[3] André, Y. **Une introduction aux motifs (motifs purs, motifs mixtes, périodes)**. Panoramas et Synthèses, **17**. Société Mathématique de France, Paris, 2004.

[4] André, Y., Kahn, B., O'Sullivan, P. *Nilpotence, radicaux et structures monoïdales*. Rend. Sem. Mat. Univ. Padova **108** (2002), 107–291.

[5] Ayoub, J., *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique*. Preprint (2006) K-theory preprint archive, http://www.math.uiuc.edu/K-theory/0761.

[6] Bloch, S. and Lichtenbaum, S., *A spectral sequence for motivic cohomology*, preprint (1995).

[7] Cisinski, D.-C. and Déglise, F. *Triangulated categories of motives* preprint, 2007.

[8] Deligne, P. and Goncharov, A., *Groupes fondamentaux motiviques de Tate mixte*. Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 1, 1–56.

[9] Dold, A. and Thom, R., *Quasifaserungen und unendliche symmetrische Produkte*. Ann. of Math. (2) **67** 1958 239–281.

[10] **The Handbook of K-theory**, vol. I, part II. E. Friedlander, D. Grayson, eds. Springer Verlag 2005.

[11] Friedlander, E. and Suslin, A., *The spectral sequence relating algebraic K-theory to motivic cohomology*. Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 6, 773–875.

[12] Dwyer, W. and Friedlander, E., *Algebraic and etale K-theory*. Trans. Amer. Math. Soc. **292** (1985), no. 1, 247–280.

[13] Geisser, T. and Levine, M., *The Bloch-Kato conjecture and a theorem of Suslin-Voevodsky*. J. Reine Angew. Math. **530** (2001), 55–103.

[14] Goerss, P. and Jardine, J., *Localization theories for simplicial presheaves*. Canad. J. Math. **50** (1998), no. 5, 1048–1089.

[15] Huber, A. and Kahn, B., *The slice filtration and mixed Tate motives*, Compositio Math. **142** (2006), 907–936.

[16] Jardine, J., *Motivic symmetric spectra*. Doc. Math. **5** (2000), 445–553.

[17] Katzarkov, L., Panteev, T. and Toen, B., *Schematic homotopy type and non-abelian Hodge theory*. Preprint (2005), arXiv:math/0107129v5 [math.AG], http://lanl.arxiv.org/abs/math/0107129.

[18] Kahn, B. and Levine, M., *Motives of Azumaya algebras*. Preprint (2007) arXiv:0710.1614v1 [math.KT], http://lanl.arxiv.org/abs/0710.1614.

[19] Kahn, B. and Sujatha, R., *A few localisation theorems*. Preprint (2007) arXiv:math/0610828v3 [math.AG] http://lanl.arxiv.org/abs/math/0610828.

[20] Kimura, S.-I., *Chow groups are finite dimensional, in some sense*. Math. Ann. **331** (2005), no. 1, 173–201.

[21] Levine, M., *Comparison of cobordism theories*. Preprint (2008). http://www.math.neu.edu/∼levine/publ/Publ.html.

[22] Levine, M. *The homotopy coniveau tower* J Topology **1** (2008) 217–267.

[23] Levine, M. and Morel, F., **Algebraic cobordism**. Springer Monographs in Mathematics. Springer, Berlin, 2007.

[24] Levine, M. and Pandharipande, R., *Algebraic cobordism revisited.* Preprint (2006), arXiv:math/0605196v1 [math.AG], http://lanl.arxiv.org/abs/math/0605196

[25] Maulik, D., Nekrasov, N., Okounkov, A. and Pandharipande, R., *Gromov-Witten theory and Donaldson-Thomas theory I*, Compos. Math. 142 (2006), no. 5, 1263–1285.

[26] Maulik, D., Nekrasov, N., Okounkov, A. and Pandharipande, R., *Gromov-Witten theory and Donaldson-Thomas theory II*, Compos. Math. 142 (2006), no. 5, 1286–1304.

[27] Mazza, C., Voevodsky, V. and Weibel, C., **Lecture notes on motivic cohomology**, Clay Monographs in Math. **2**. Amer. Math. Soc. 2006.

[28] Merkurjev, A. and Suslin, A., *K-cohomology of Severi-Brauer varieties and the norm residue homomorphism.* Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), no. 5, 1011–1046, 1135–1136.

[29] Milne, J., **Etale cohomology**. Princeton Mathematical Series, **33**. Princeton University Press, Princeton, N.J., 1980.

[30] Milnor, J., *Algebraic K-theory and quadratic forms.* Invent. Math. **9** (1969/1970), 318–344.

[31] Morel, F., $\mathbb{A}^1$-algebraic topology. International Congress of Mathematicians. Vol. II, 1035–1059, Eur. Math. Soc., Zrich, 2006.

[32] Morel, F., *Some basic properties of the the stable homotopy category of schemes.* Preprint (2005).

[33] Morel, F., *An introduction to $\mathbb{A}^1$-homotopy theory.* Contemporary developments in algebraic K-theory, 357–441, ICTP Lect. Notes, XV, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.

[34] Morel, F., *On the motivic π_0 of the sphere spectrum.* Axiomatic, enriched and motivic homotopy theory, 219–260, NATO Sci. Ser. II Math. Phys. Chem., **131**, Kluwer Acad. Publ., Dordrecht, 2004.

[35] Morel, F., **Théorie homotopique des schémas**. Astérisque **256** (1999).

[36] Morel, F. and Voevodsky, V., **$\mathbb{A}^1$-homotopy theory of schemes**. Inst. Hautes Études Sci. Publ. Math. **90** (1999), 45–143 (2001).

[37] Nesterenko, Y. and Suslin, A., *Homology of the general linear group over a local ring, and Milnor's K-theory.*Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 1, 121–146.

[38] Pelaez-Menaldo, J.P., *Multiplicative properties of the slice filtration*. Ph.D. thesis, Northeastern University (2008).

[39] Quillen, D. G., *Elementary proofs of some results of cobordism theory using Steenrod operations*, Advances in Math. **7** (1971) 29–56.

[40] Röndigs, O. and Østvær, P. A., *Motives and modules over motivic cohomology*. C. R. Math. Acad. Sci. Paris **342** (2006), no. 10, 751–754.

[41] Rost, M., *Chain lemma for symbols*. Chain Lemma page. http://www.math.uni-bielefeld.de/∼rost/chain-lemma.html.

[42] Serre, J.-P., **Algèbre locale. Multiplicités**. Cours au Collège de France, 1957–1958, rédigè par Pierre Gabriel. Seconde édition, 1965. Lecture Notes in Mathematics, **11** Springer-Verlag, Berlin-New York 1965 vii+188 pp.

[43] Spanier, E., **Algebraic topology**. Corrected reprint. Springer-Verlag, New York-Berlin, 1981. xvi+528

[44] Suslin, A., *Algebraic K-theory and motivic cohomology*. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zrich, 1994), 342–351, Birkhäuser, Basel, 1995.

[45] Suslin, A. and Joukhovitski, S., *Norm varieties*. J. Pure Appl. Algebra **206** (2006), no. 1-2, 245–276.

[46] Suslin, A. and Voevodsky, V., *Bloch-Kato conjecture and motivic cohomology with finite coefficients*. The arithmetic and geometry of algebraic cycles (Banff, AB, 1998), 117–189, NATO Sci. Ser. C Math. Phys. Sci., **548**, Kluwer Acad. Publ., Dordrecht, 2000.

[47] Tate, J., *Relations between K_2 and Galois cohomology*. Invent. Math. **36** (1976), 257–274.

[48] Totaro, B., *Milnor K-theory is the simplest part of algebraic K-theory*. K-Theory **6** (1992), no. 2, 177–189.

[49] Voevodsky, V., *Simplicial radditive functors*. Preprint (2007). K-theory preprint archives, http://www.math.uiuc.edu/K-theory/0863/

[50] Voevodsky, V., *Motivic Eilenberg-MacLane spaces*. Preprint (2007). K-theory preprint archives, http://www.math.uiuc.edu/K-theory/0864/

[51] Voevodsky, V., *On the zero slice of the sphere spectrum*. Tr. Mat. Inst. Steklova **246** (2004), Algebr. Geom. Metody, Svyazi i Prilozh., 106–115; translation in Proc. Steklov Inst. Math. 2004, **3** (246), 93–102.

[52] Voevodsky, V., *Motivic cohomology with $\mathbb{Z}/2$-coefficients*. Publ. Math. Inst. Hautes Études Sci. **98** (2003), 59–104.

[53] Voevodsky, V., *Reduced power operations in motivic cohomology*. Publ. Math. Inst. Hautes Études Sci. **98** (2003), 1–57.

[54] Voevodsky, V., *A possible new approach to the motivic spectral sequence for algebraic K-theory*. Recent progress in homotopy theory (Baltimore, MD, 2000), 371–379, Contemp. Math., **293**, Amer. Math. Soc., Providence, RI, 2002.

[55] Voevodsky, V., *$\mathbb{A}^1$-homotopy theory*. Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998). Doc. Math. 1998, Extra Vol. I, 579–604.

[56] Voevodsky, V., Röndigs, O. and Østvær, Paul Arne, *Voevodsky's Nordfjordeid lectures*, motivic homotopy theory. Motivic homotopy theory, 147–221, Universitext, Springer, Berlin, 2007.

[57] Voevodsky, V. Suslin, A. and Friedlander, E. M., **Cycles, transfers, and motivic homology theories**. Annals of Mathematics Studies, **143**. Princeton University Press, Princeton, NJ, 2000.

[58] Vistoli, A., *Grothendieck topologies, fibered categories and descent theory*. Fundamental algebraic geometry, 1–104, Math. Surveys Monogr., **123**, Amer. Math. Soc., Providence, RI, 2005.

[59] Weibel, C., *Patching the Norm Residue Isomorphism Theorem*. Preprint (2007) K-theory preprint archives,
http://www.math.uiuc.edu/K-theory/0844/

Marc Levine
Department of Mathematics
Northeastern University
Boston, MA 02115
U.S.A.
e-mail: `marc@neu.edu`