Art Gallery and Illumination Problems

Jorge Urrutia*

Department of Computer Science

University of Ottawa

Ottawa, Ontario, Canada

E-mail: `jorge@csi.uottawa.ca`

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Abstract

This is the preliminary version of a chapter that will appear in *Handbook on Computational Geometry*, edited by J.-R. Sack and J. Urrutia.

In 1973, Victor Klee posed the following question: How many guards are necessary, and how many are sufficient to patrol the paintings and works of art in an art gallery with n walls? This wonderfully naïve question of combinatorial geometry has, since its formulation, stimulated a plethora of papers, surveys and a book, most of them written in the last fifteen years. The first result in this area, due to V. Chvátal, asserts that $\lfloor \frac{n}{3} \rfloor$ guards are occasionally necessary and always sufficient to guard an art gallery represented by a simple polygon with n vertices. Since Chvátal's result, numerous variations on the art gallery problem have been studied, including mobile guards, guards with limited visibility or mobility, illumination of families of convex sets on the plane, guarding of rectilinear polygons, and others. In this paper, we survey most of these results.

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Contents

1 Introduction

Illumination problems have always been a popular topic of study in Mathematics. For example, it is well known that the boundary of any smooth compact convex set on the plane can always be illuminated using three light sources. One famous—and until recently open—problem on illumination is attributed to Ernst Strauss (see E. G. Strauss and V. Klee [80]), who in the early fifties posed the following problem:

Suppose that we live in a two-dimensional room whose walls form a simple closed polygon P and each wall is a mirror.

1. Is it true that if we place a light at any point of P, all of P will be illuminated using reflected rays as well as direct rays?

2. Is there necessarily a point from which a single light source will illuminate the entire room using reflected rays as well as direct rays?

The first part of Strauss's problem was recently proved to be false by G. W. Tokarsky [119]. Tokarsky's proof is surprisingly simple, using basic concepts of geometry that are easily understandable. We refer the interested reader to Tokarsky's original manuscript which is clearly written and a pleasure to read. The second part of Strauss's conjecture, though, remains open. It would be nice if a "simple" proof for it could be obtained.

More closely related to our topic of interest here is a question posed by V. Klee during a conference in Stanford in August 1976. Klee's question was: *How many guards are always sufficient to guard any polygon with n vertices*? Soon after, V. Chvátal established what has become known as *Chvátal's Art Gallery Theorem*, namely: that $\lfloor \frac{n}{3} \rfloor$ guards are always sufficient and occasionally necessary to cover a simple polygon with n vertices [25].

Since the publication of this original result, a tremendous amount of research on illumination or Art Gallery problems has been carried out by mathematicians and computer scientists. In 1987, J. O'Rourke [96] published *Art Gallery Theorems and Algorithms*, the first book dedicated solely to the study of illumination problems of polygons on the plane. The publication of this book further fueled the study of Art Gallery type problems, and many variations to the original Art Gallery Theorem have since been studied. In 1992, T. Shermer [107] published a thorough survey paper on Art Gallery problems. Since then a large number of papers in this area have appeared, and some important open problems have been solved. In this survey, we try to cover most of the results published to date in Art Gallery or Illumination Theorems. Visibility graphs, studied in O'Rourke's book and Shermer's survey, will not be covered in this survey; they are surveyed in another paper in this volume.

2 Basic terminology

A polygon P is an ordered sequence of points $p_1, \ldots, p_n$, $n \geq 3$, called the vertices of P together with the set of line segments joining p_i to p_{i+1}, $i = 1, \ldots, n-1$ and p_n to p_1, called the edges of P. P is called simple if any two non-consecutive edges do not intersect. A simple polygon divides the plane into two regions, an unbounded one called the exterior region and a bounded one, the interior. Henceforth to simplify our presentation, the term polygon will be used to denote simple polygons together with their interior.

A simple polygon is called *orthogonal* if all its edges are parallel to either the x-axis or the y-axis.

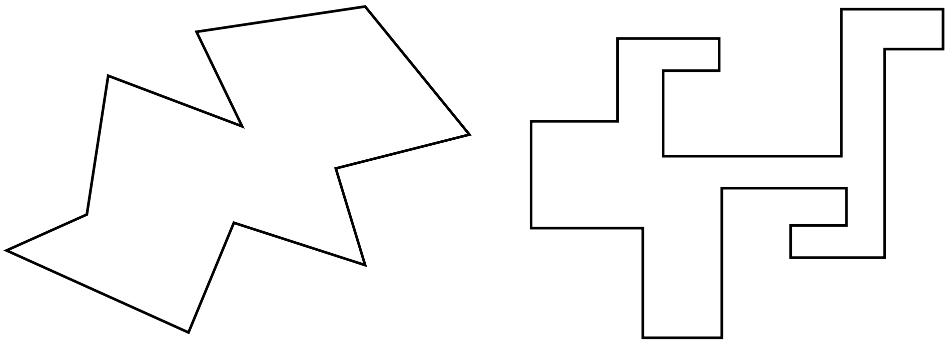

Figure 1: A simple and an orthogonal polygon.

Given two points p and q of a polygon P, we say that p is *visible* from q if the line segment joining p to q is totally contained in P.

A collection H of points of P *illuminates* or *guards* P if every point u of P is visible from a point p in H. The term *illuminates* follows the notion that if we place a light source that emits light *in all directions* that falls on each element of H, P is totally illuminated. The use of the term *guard* follows the notion that if we station a guard at each element of H, all of P is guarded. To illuminate the orthogonal polygon in Figure 1 we need four lights. We remark at this point that the terms *illumination* and *guarding* will be used interchangeably in this manuscript. Our choice of whether to "guard" or "illuminate" an object depends mainly on the term used in the original paper in which a particular result was proved.

A triangulation T of a polygon P is a partitioning of P into a set of triangles with pairwise interiors in such a way that the edges of those triangles are either edges or diagonals of P joining pairs of vertices. It is easy to see that any triangulation of a polygon P with n vertices contains exactly $n-2$ triangles.

Triangulations of polygons play a central role in the study of Art Gallery problems. The problem of finding efficient algorithms to triangulate polygons has received much attention in Computational Geometry. In 1978, Garey, Johnson, Preparata and Tarjan [57] obtained the first $O(n \ln n)$ time triangulation algorithm. This result was improved in 1988 by Tarjan and van Wyk to $O(n \ln \ln n)$ [118]. Finally in 1990, Chazelle [18] obtained an optimal linear time algorithm to solve this problem. At the time some of the algorithmic results surveyed here were published, they contained a logarithmic or double logarithmic factor introduced by the use of a triangulation algorithm. In these cases, these factors have been dropped.

A graph $G(V, E)$ consists of a set of elements V called the vertices of $G(V, E)$, together with a set E of pairs of vertices of $G(V, E)$ called the edges of $G(V, E)$. Two vertices u and v of $G(V,E)$ are called adjacent if the pair $\{u,v\}$ is an element of E. A graph $G(V,E)$ is planar if it can be drawn on the plane in such a way that its vertices are represented by points on the plane, and each edge $\{u,v\}$ of $G(V,E)$ is represented by a simple curve joining points representing u and v. Moreover two edges of $G(V,E)$ may only intersect at their endpoints.

A path of $G(V,E)$ is a sequence of distinct vertices $v_1,\ldots,v_k$ such that v_i and v_{i+1} are adjacent in $G(V,E)$, $i=1,\ldots,k-1$, $k\geq 2$. A cycle of $G(V,E)$ is a path $v_1,\ldots,v_k$ together with the edge $\{v_k,v_1\}$, $k\geq 3$.

A graph $G(V,E)$ is called connected if for every pair of vertices u and v of $G(V,E)$, there exists a path $u=v_1,\ldots,v_k=v$ starting at u and ending at v, otherwise $G(V,E)$ is called disconnected. A graph $G(V,E)$ is called *k-connected* if, when we remove any m vertices of $G(V,E)$ together with the edges adjacent to them, we obtain a connected graph, $m<k$. From now on, unless necessary, we shall refer to a graph $G(V,E)$ only as G.

Two points u and v in a polygon P are called *visible* if the line segment joining them is contained in P. The visibility graph $VG(P)$ of P is the graph whose vertex set is the set of vertices of P, two vertices u and v being adjacent in $VG(P)$ if they are visible in P. Visibility graphs were first introduced by Avis and ElGindy [5].

Avis and ElGindy gave an optimal (worst case) $O(n^2)$ time algorithm to compute the visibility graph of a polygon. Their result was later improved by Hershberger [69], who gave an $O(|E|)$ time algorithm, where $|E|$ is the number of edges of the visibility graph of P. One of the most important open problems in Computational Geometry is that of characterizing and recognizing visibility graphs.

Two results in Matching Theory will be central to our results, and for completeness we introduce them now. Given a graph G, a *matching* M of G is a subset of edges of G such that no two edges of M have a common vertex. A matching M of G is called *perfect* if each vertex of G is a vertex of an edge in M.

Given a graph G and a subset S of the vertices of G, we define $Odd(S)$ to be the number of components of $G-S$ with an odd number of vertices. The following result due to Tutte provides necessary and sufficient conditions for the existence of perfect matchings:

Theorem 2.1 *(Tutte [120]): A graph G has a perfect matching iff for every subset S of $V(G)$, $Odd(G-S)\leq |S|$.*

For the case of planar graphs, we will use the following result by T. Nishizeki extensively:

Theorem 2.2 *(Nishizeki [91]): Any planar 2-connected graph G with $n\geq 14$ vertices and minimum vertex degree greater than or equal to 3 has a maximum matching with at least $\lfloor\frac{n+4}{3}\rfloor$ edges. When $n\leq 14$, the number of edges in such a matching is $\lfloor\frac{n}{2}\rfloor$.*

The proofs of these results are not of concern in this work; the interested reader can find them in many books on graph theory.

3 Four theorems in Art Galleries and Illumination

The study of Art Gallery problems deals mainly with illumination of polygons, families of convex sets and most recently, floodlight illumination problems; that is illumination problems using light sources with restricted angle of illumination. To illustrate the main techniques used in this area of research, in this section we give four results with complete proofs.

3.1 Chvátal's classical Art Galley theorem

We start with the proof of Chvátal's Art Gallery Theorem. The proof presented here is due to S. Fisk [53].

Theorem 3.1 $\lfloor \frac{n}{3} \rfloor$ *stationary guards are always sufficient and occasionally necessary to illuminate a polygonal art gallery with n vertices.*

Proof: Consider an arbitrary simple polygon P with n vertices. Obtain a triangulation T of P by adding $n-2$ interior diagonals (See Figure 2.)

It is easy to see that we can color the vertices of P using three colors $\{1, 2, 3\}$ such that any two vertices joined by an edge of P or a diagonal of T receive different colors. This partitions the vertex set of P into three chromatic classes C_1, C_2 and C_3. Since the vertices of each triangle of T receive different colors, each chromatic class guards P. Place a guard at each vertex of the smallest chromatic class and our result follows.

To see that $\lfloor \frac{n}{3} \rfloor$ guards are sometimes needed, consider the comb polygon $Comb_m$ with $n = 3m$ vertices presented in Figure 2. It is easy to see that to guard P_m we need at least m guards. ∎

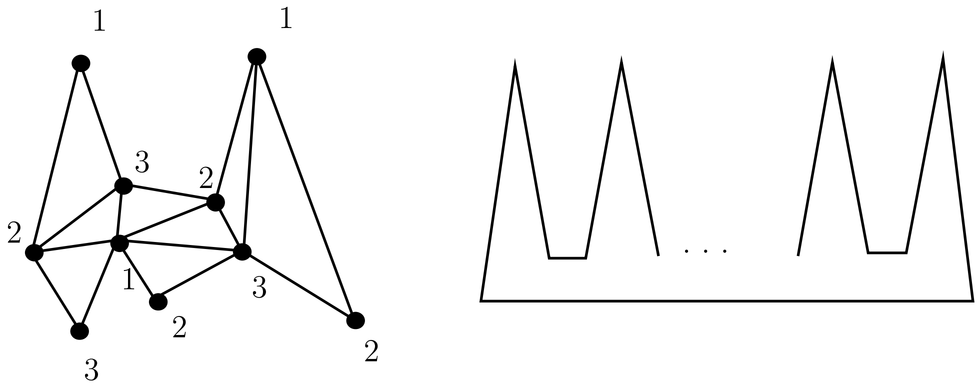

Figure 2: Illustration of proof of Chvátal's Art Gallery Theorem.

3.2 Guarding traditional art galleries

In the classical Art Gallery Theorem, an art gallery is a simple polygon on the plane. In a more realistic setting, a *traditional art gallery* is housed in a rectangular building subdivided into rectangular rooms. Assume that any two adjacent rooms have a door connecting them. (See Figure 3.)

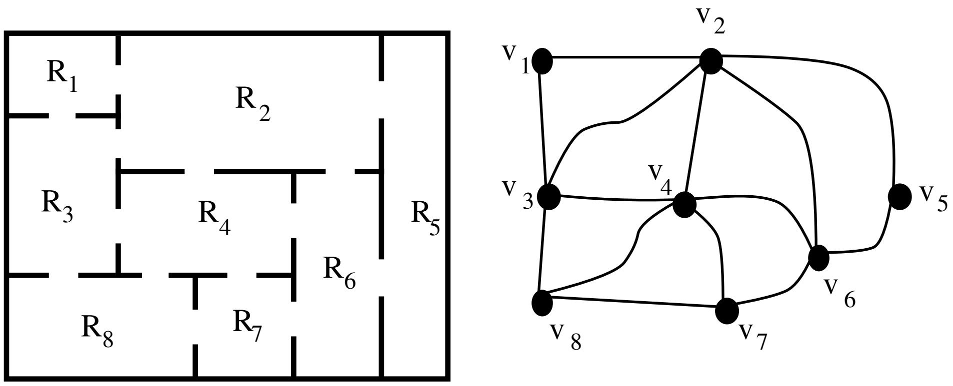

Figure 3: Traditional art gallery and its dual graph.

How many guards need to be stationed in the gallery so as to guard all its rooms? Notice that if a guard is stationed at a door connecting two rooms, he will be able to guard both rooms at once, and since no guard can guard three rooms, it follows that if the art gallery has n rooms we need at least $\lceil \frac{n}{2} \rceil$ guards. We now prove:

Theorem 3.2 *[38] Any rectangular art gallery with n rooms can be guarded with exactly $\lceil \frac{n}{2} \rceil$ guards.*

Proof: Given a rectangular art gallery T with n rooms $R_1, \ldots, R_n$, we can associate to it a dual graph $G(T)$ by representing each room R_i of T by a vertex v_i in $G(T)$, two vertices being adjacent if their corresponding rectangles share a line segment in their common boundary. See Figure 3.

Notice that the outermost boundary of the union of the rectangles corresponding to the vertices of any connected subgraph of $G(T)$ form an orthogonal polygon.

We now show that if $G(T)$ has an even number of vertices, $G(T)$ has a perfect matching M. This suffices to prove our result since our $\lceil \frac{n}{2} \rceil$ points can now be chosen using M as follows: for every edge $\{v_i, v_j\}$ of $G(T)$ in M station a guard at the door connecting R_i and R_j. Clearly these guards will cover all of the subrectangles of T. The case when $G(T)$ has an odd number of vertices follows by subdividing any room of T into two.

Let us now assume that $G(T)$ has an even number of vertices. To prove that $G(T)$ has a perfect matching, we will show that it satisfies the conditions for the existence of a perfect matching stated in Tutte's Theorem; namely that for any subset S of the vertices of $G(T)$ the number of odd components of $G(T) - S$ does not exceed $|S|$. Let

k be the number of connected components of $G(T)-S$. Each connected component C_i of $G(T)-S$ is represented by an orthogonal subpolygon P_i of T. Each such polygon has at least four corner points, and thus the total number of corner points generated by the k components in $G(T)-S$ is at least $4k$.

The next observation is essential to our proof: When a rectangle represented by a point in S is now replaced, at most four corner points will disappear.

Once all rectangles in S are replaced, all the corner points generated by the components of $G(T)-S$ will disappear, except for the four corner points of T. It follows that $k \leq |S|+1$. The reader may verify that if $k=|S|+1$, then at least one of the components of $G(T)-S$ is even. ∎

3.3 Illuminating families of convex sets

A folklore result in mathematics asserts that to illuminate the boundary of a compact convex set S on the plane, three light sources always suffice.

The following problem was first studied by Fejes Tóth [52]. Let $F=\{S_1,\ldots,S_n\}$ be a family of n disjoint compact convex sets on the plane. How many light sources located in the complement of $S_1 \cup \ldots \cup S_n$ are always sufficient to completely illuminate the boundaries of the elements of F?

In this section we prove:

Theorem 3.3 *[52] For any family $F=\{S_1,\ldots,S_n\}$ of n disjoint compact convex sets, $4n-7$ lights located in the complement of $S_1 \cup \ldots \cup S_n$ are always sufficient and occasionally necessary to illuminate the boundaries of the elements of F.*

Proof: Construct a family $T=\{T_1,\ldots,T_n\}$ of n strictly convex such that:

1. S_i is contained in T_i, $i=1,\ldots,n$.
2. The interiors of T_i, $i=1,\ldots,n$ are disjoint.
3. The number of tangencies between the elements of T is maximized.

Suppose that T_i is tangent to $T_{s(j)}$, $j=1,\ldots,m$. Consider the lines $l_{i,s(j)}$ tangent to T_i at the points in which T_i intersects $T_{s(j)}$, $j=1,\ldots,m$. For each i, $T_{s(1)},\ldots,T_{s(m)}$ define a polygonal region P_i. Two cases arise:

1. P_i is a bounded polygonal region. In this case, place a light at each vertex of P_i.
2. P_i is an unbounded polygonal region. For this case, place a light at each vertex of P_i, and one more at each of two semilines of P_i far enough from S_i.

It is easy to see that these lights illuminate S_i, $i=1,\ldots,n$ and that each of the lines defined above is assigned exactly two lights. See Figure 4.

Construct a planar graph G with vertex set $\{T_1,\ldots,T_n\}$ such that T_i and T_j are adjacent if they are tangent. It is clear that this graph is planar, and thus it has at most $3n-6$ edges. Notice that there is a one to one correspondence between the edges

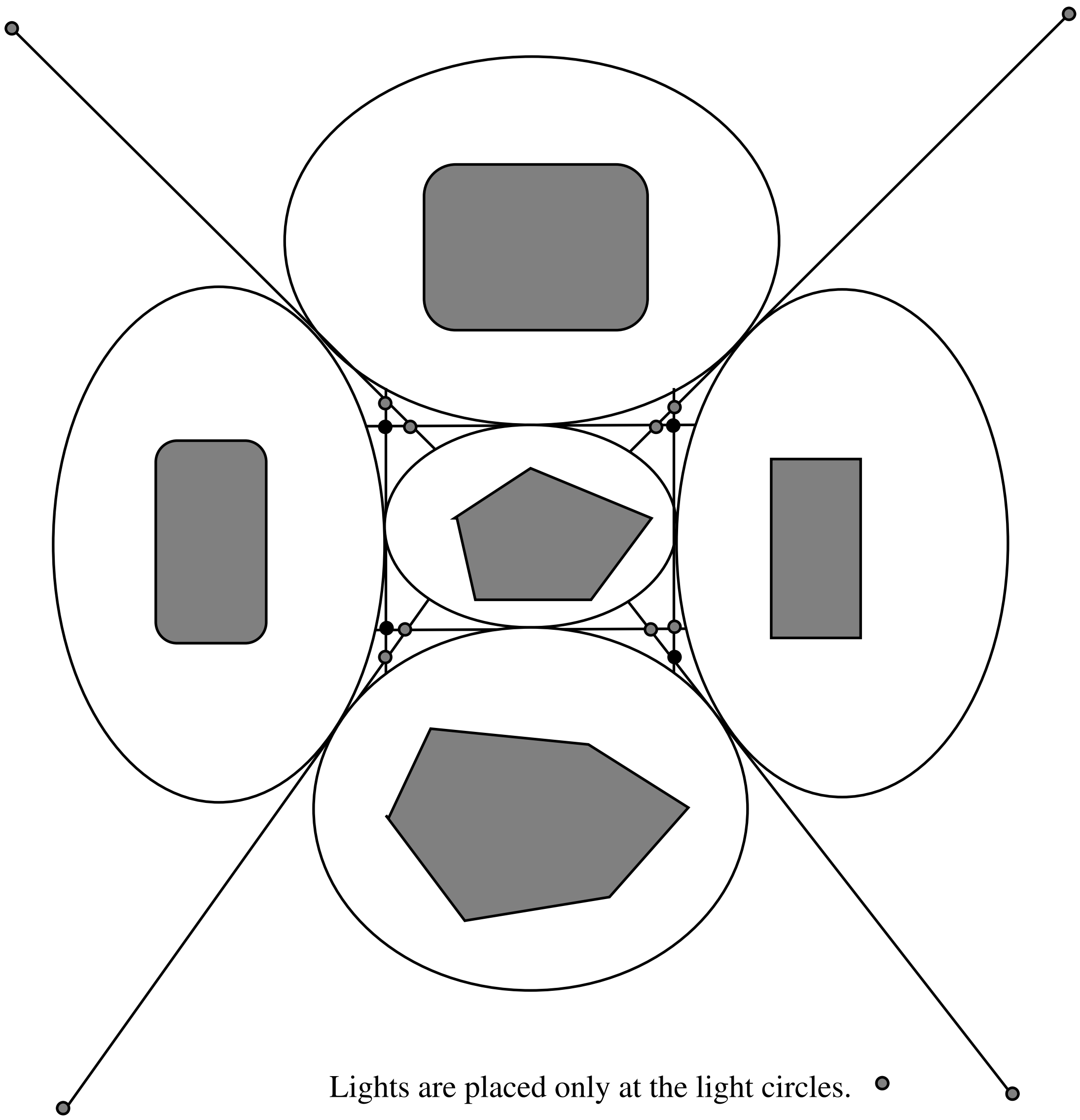

Figure 4: Illustration of proof of theorem on illumination families of convex sets.

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of G and the set of tangents generated by $T = \{T_1, \ldots, T_n\}$. It now follows that the number of lights used so far is at most $2(3n-6)$.

Let H be the complement of the union of T_i, $i = 1, \ldots, n$. H consists of an unbounded face and a number of bounded ones. It is easy to see that the number of lights needed to illuminate $S_1, \ldots, S_n$ is maximized when all of the faces of H, including the outer one, are bounded by exactly three elements of T. We analyze this case only, and leave the rest to the reader. We notice now that each face of H contains exactly 3 lights. Note, however, that if for each face of H except the outer face, we remove one of these three lights, the remaining lights still illuminate all the elements of F; see Figure 4. It is now easy to see that $4n-7$ lights remain, and our result follows. ∎

3.4 A floodlight illumination problem

In the previous problems, we have assumed that the light sources emit light in all directions, or that the guards can patrol around themselves in all directions. We now present an illumination problem in which the light sources have a restricted angle of illumination. We call such light sources *floodlights*. Thus for the rest of this paper, a floodlight f_i is a source of light located at a point p of the plane, called its apex; f_i illuminates only within a positive angle of illumination α_i, and can be rotated around its apex. We study the following problem due to J. Urrutia:

Problem 3.1 The 3-floodlight illumination problem*: Let $\alpha_1 + \alpha_2 + \alpha_3 = \pi$ and consider any convex polygon P. Can we place three floodlights of sizes at most $\alpha_1, \alpha_2, \alpha_3$, at most one per vertex, in such a way that P is completely illuminated?*

We now show that the Three Floodlight Illumination Problem always has a positive solution. Clearly our result is true if P has 3 vertices. Consider any convex polygon P with at least four vertices and suppose that $\alpha_1 \leq \alpha_2 \leq \alpha_3$. Notice first that $\alpha_2 < \frac{\pi}{2}$, and that since P has at least four vertices, the interior angle at one vertex v of P is at least $\frac{\pi}{2}$. Find a triangle T with internal angles α_1, α_2, and α_3 such that the vertex of T with angle α_2 lies on v, and the other vertices of T lie on two points x and y on the boundary of P. Suppose that x and y lie on different edges, say e_1 and e_3 of P. (The case when they lie on the same edge will be left to the reader.) See Figure 5.

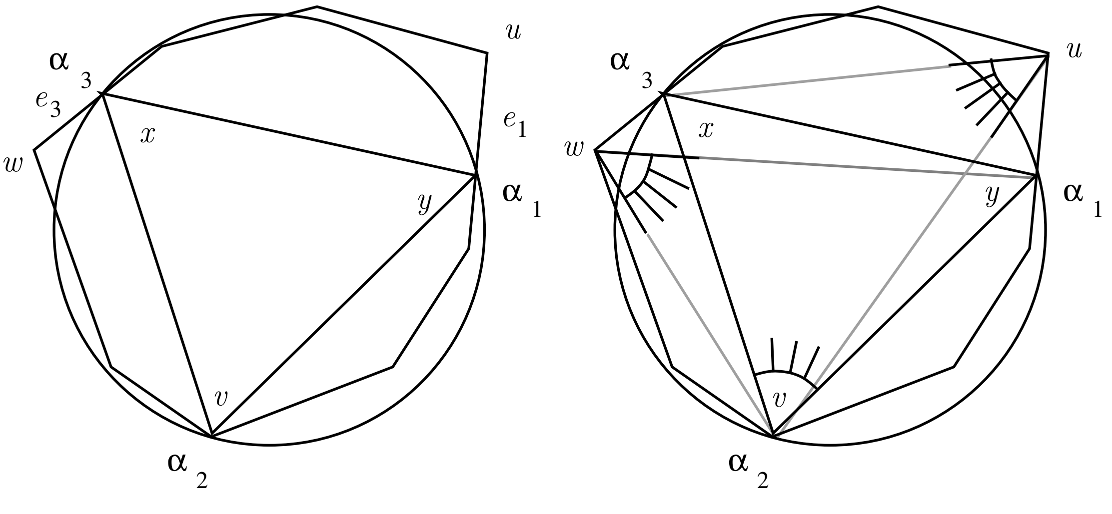


Figure 5: Illustration of proof of the Three Floodlight Problem.

Place a floodlight f_2 with angle of illumination α_2 at v, illuminating T. Consider the circle C passing through the vertices of T. It is easy to see that at least one vertex of each of e_1 and e_3 is not contained in the interior of C. Let u and w be these end points. Two cases arise:

1. $u \neq w$. Place floodlights f_1 and f_3 at u and w illuminating the angular region determined by v, u, x and v, w, y. Since u and w are not contained in the interior of C, the angles of illumination of f_1 and f_3 are at most α_1 and α_3 respectively. Since f_1, f_2, and f_3 illuminate P our result follows.

2. $u = w$. By considering the tangents to C at x and y it is easy to verify that the angle α of P at u is at most $\pi - 2\alpha_2$ which is less than or equal to $\alpha_3 = \pi - (\alpha_1 + \alpha_2)$. Place a floodlight of size α at u. This illuminates P. Our result follows.

4 Variations on the Art Gallery Problem

4.1 Edge guards, vertex guards, etc.

There are several types of restrictions that may be imposed on the guards used to cover a polygon P. In some cases, we may want them to be stationed at vertices of P. In other cases, they can be located anywhere within P, or patrol along diagonals or edges of P. We now list all the different types of guards or light sources studied in the literature.

Point guards These are guards that can be located anywhere in the polygon to be guarded.

Vertex guards In this case, the positions of the guards are restricted to vertices of a polygon.

The distinction between point and vertex guards is important. In many of our results, the bounds obtained for these two types of guards are different.

Edge guards Edge guards were introduced by Toussaint in 1981. His original motivation was that of allowing a guard to move along the edges of a polygon. A point q can be considered guarded if it is visible from some point in the path of a guard. Alternately, we could think of the illumination problem of a polygon P in which we are allowed to place "fluorescent" lights along the edges of P; each fluorescent light covers the whole length of an edge of P. Within this setting, our problem becomes: How many "fluorescent" lights are needed to illuminate a polygon with n vertices?

Mobile guards O'Rourke [97] introduced this variation in which the guards are allowed to move along closed line segments totally contained in a polygon P.

Vertex and point floodlights We also distinguish between vertex floodlights, located at vertices of a polygon, and point floodlights, which can be located anywhere in the polygon to be guarded. Floodlights were introduced by J. Urrutia in 1990 at a workshop on illumination in Bell Airs, Barbados. The motivation for this type of guard is that many guarding and broadcasting devices have a limited range of visibility.

4.2 The complexity of the Art Gallery Theorem

Fisk's proof of Chvátal's Theorem to find $\lfloor \frac{n}{3} \rfloor$ guards to cover a polygon leads in a natural way to an efficient algorithm. First triangulate the polygon P to be guarded, then 3-color the resulting graph and station a guard at all the vertices of the smallest chromatic class. This was noticed first by Avis and Toussaint [7] who gave an $O(n \ln n)$ time algorithm to solve the Art Gallery Theorem. Their result was later improved when Tarjan and van Wyk [118] obtained an $O(n \ln \ln n)$ time algorithm to triangulate polygons. The complexity of the Art Gallery Theorem was finally settled when Chazelle obtained a linear time triangulation algorithm [18].

On the other hand, the problem of finding the *minimum* number of guards needed to guard a polygon is much harder. Lee and Lin [82] proved:

Theorem 4.1 *The minimum vertex guard problem for polygons is NP-hard.*

Their proof is based on a reduction of the 3-satisfiability problem [56]. In the same paper, they show that the *minimum edge* and *point guard* problems are also NP-hard.

One approach that has been neglected in the study of Art Gallery problems is that of finding algorithms that obtain approximate solutions in terms of optimal ones. To our knowledge, there is only one paper on this subject (Ghosh [64]). He obtained an algorithm that, given a polygon P with n vertices, finds in $O(n^5 \ln n)$ time a vertex guard set that has at most $O(\ln n)$ times the minimum number of vertex guards needed to guard P.

For orthogonal polygons, Schuchardt and Hecker [104] recently proved:

Theorem 4.2 *The minimum vertex- and point-guard problems for orthogonal polygons is NP-hard.*

4.3 Orthogonal polygons

Guarding problems for orthogonal polygons have received much attention. This is perhaps because most real buildings are "orthogonal", and thus they are a better model for potential applications of our results. From a mathematical point of view, their inherent structure allows us to obtain very interesting and aesthetic results. The first major result here is due to Kahn, Klawe and Kleitman [79]. They proved:

Theorem 4.3 *Any orthogonal polygon with n vertices can always be illuminated with $\lfloor \frac{n}{4} \rfloor$ vertex guards. $\lfloor \frac{n}{4} \rfloor$ guards are sometimes necessary.*

Kahn, Klawe and Kleitman's proof was based on a similar technique to that used by Fisk. The main idea of their proof is to partition an orthogonal polygon into convex quadrilaterals. The internal diagonals of each of these quadrilaterals are then added, and the graph thus obtained four-vertex colored.

Kahn, Klawe and Kleitman's result provided an incentive to study the problem of decomposing a rectilinear polygon into convex quadrilaterals. The first $O(n)$ algorithm to achieve this was obtained by Sack [101] (see also [103]). A different algorithm

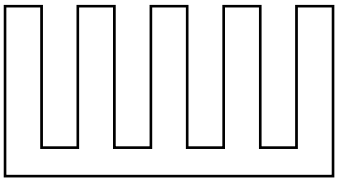


Figure 6: A polygon that requires $\lfloor \frac{n}{4} \rfloor$ guards.

was later obtained by Lubiw [87]. Related results can also be found in Sack and Toussaint [102].

A linear time algorithm to guard orthogonal polygons using $\lfloor \frac{n}{4} \rfloor$ point guards, some of which may be located in the interior of the polygon, was then given by Edelsbrunner, O'Rourke and Welzl [43]. Their result is based on an L-shaped partitioning of orthogonal polygons.

In 1986, E. Györy gave another proof of Theorem 4.3. His proof also decomposes an orthogonal polygon into a set of at most $\lfloor \frac{n}{4} \rfloor$ L-shaped polygons.

More recently, Estivill-Castro and Urrutia [47] (see also [1]) proved that $\lfloor \frac{n}{4} \rfloor$ guards are also sufficient to guard an orthogonal polygon, but in this case using orthogonal floodlights; that is guards which have an angle of vision of 90 degrees. Moreover, unlike the result of Edelsbrunner, O'Rourke and Welzl, they locate the lights on the edges of the polygon. For more details see Theorem 6.7 of this survey. They also give a linear time algorithm to solve this problem.

4.4 Polygons with holes

Given a polygon P and a set of m disjoint polygons $P_1, \ldots, P_m$ contained in the interior of P, we call the set $P - \{P_1 \cup \ldots \cup P_m\}$ a polygon with holes. In this case, we say that P has m holes.

O'Rourke [97] proved the first result on guarding polygons with holes:

Theorem 4.4 *Any polygon with n vertices and h holes can always be guarded with $\lfloor \frac{(n+2h)}{3} \rfloor$ vertex guards.*

O'Rourke's proof proceeds by eliminating the holes of a polygon P one at a time by cutting it along a line segment joining a diagonal joining two vertices of P; one a vertex of a hole, the other on the outer face of P.

It is widely believed that this bound is not tight, in fact Shermer conjectures:

Conjecture 4.1 *Any polygon with n vertices and h holes can always be guarded with $\lfloor \frac{(n+h)}{3} \rfloor$ vertex guards.*

Shermer proved his conjecture for the case $h = 1$; see [96]. However, for $h > 1$ the conjecture remains open. Substantial progress has been achieved for point gurds. Bjorling-Sachs and Souvaine [10] and Hoffmann, Kaufman, and Kriegel [73] independently proved:

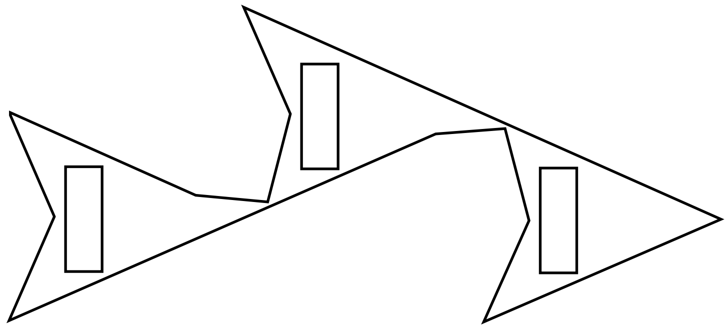


Figure 7: A polygon that requires $\lfloor \frac{n+h}{3} \rfloor$ guards.

Theorem 4.5 $\lceil \frac{(n+h)}{3} \rceil$ *point guards are always sufficient and occasionally necessary to guard any polygon with n vertices and h holes.*

In 1995, Bjorling-Sachs and Souvaine [11] gave an $O(n^2)$ time algorithm to find the position of the $\lceil \frac{(n+h)}{3} \rceil$ guards. The main idea in Bjorling-Sachs and Souvaine's paper is to connect each hole of the polygon to the exterior by cutting away a quadrilateral *channel* such that one vertex is introduced for each channel, and there is a triangle T in the remaining polygon such that any point in it sees all of the channel. This triangle is then forced to be in a triangulation of the remaining polygon, and using Fisk's proof of Chvatal's Theorem, we place a guard at a vertex of T. This will cover the channel, and the result follows.

4.5 Orthogonal polygons with holes

In 1982, O'Rourke [96] proved that any orthogonal polygon with n vertices and h holes can always be guarded with $\lfloor \frac{n+2h}{4} \rfloor$ vertex guards. He conjectured that $\lfloor \frac{n}{4} \rfloor$ vertex guards are always sufficient to guard them. Aggarwal [4] was able to verify this conjecture for $h = 1, 2$. This conjecture remained open until 1990, when F. Hoffmann [72] proved:

Theorem 4.6 $\lfloor \frac{n}{4} \rfloor$ *point guards are always sufficient to guard any orthogonal polygon with n vertices and h holes.*

For vertex guards, the best known upper bound, due again to O'Rourke, remains at $\lfloor \frac{n+2h}{4} \rfloor$. When h is "large", an improvement was also achieved by Hoffman and Kriegel [74]. They proved:

Theorem 4.7 $\lfloor \frac{n}{3} \rfloor$ *vertex guards are always sufficient to guard an orthogonal polygon with holes.*

The main idea behind their proof is the following result which is interesting in its own right:

Theorem 4.8 *Any planar bipartite graph can be completed to a maximal planar graph which is 3-vertex colorable.*

To prove their result, they first partition the orthogonal polygon into convex quadrilaterals, and then triangulate the resulting graph in such a way that every vertex has even degree. Then they use a theorem by Whitney that states that any planar triangulated graph is 3-vertex colorable iff all its vertices have even degree. Their result leads to an $O(n^2)$ time algorithm.

For the lower bound, Hoffman found a family of polygons with holes that require $\lfloor \frac{2n}{7} \rfloor$ vertex guards. This disproved an earlier conjecture by Aggarwal [4] that $\lfloor \frac{3n}{11} \rfloor$ were always sufficient. Hoffman conjectures:

Conjecture 4.2 $\lfloor \frac{2n}{7} \rfloor$ *vertex guards are always sufficient to guard any orthogonal polygon with holes.*

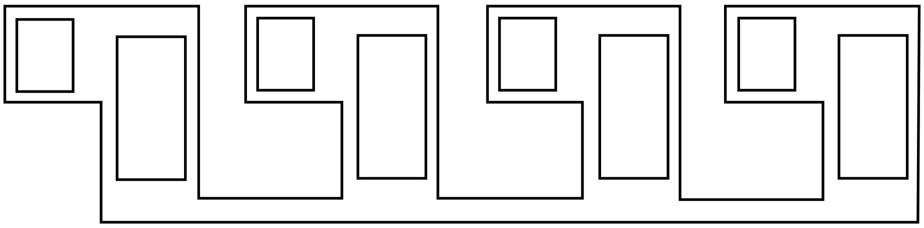


Figure 8: Hoffman's polygon requiring $\lfloor \frac{2n}{7} \rfloor$ vertex guards.

4.6 Edge and mobile guards

In 1981, Toussaint asked the question of determining the minimum number of edge guards required to guard any polygon with n vertices. He conjectured:

Conjecture 4.3 *Except for a few polygons,* $\lfloor \frac{n}{4} \rfloor$ *edge guards are always sufficient to guard any polygon with* n *vertices.*

In the next figure, we show a typical polygon that requires $\lfloor \frac{n}{4} \rfloor$ edge guards, as well as the only two known counterexamples to this conjecture due to Paige and Shermer. See Figure 9.

This conjecture remains open. The first positive result in this direction is due to O'Rourke [97]. By allowing the guards to move along diagonals joining vertices of P, i.e. using *mobile guards*, he was able to prove:

Theorem 4.9 $\lfloor \frac{n}{4} \rfloor$ mobile guards *are always sufficient and occasionally necessary to guard any polygon with* n *vertices.*

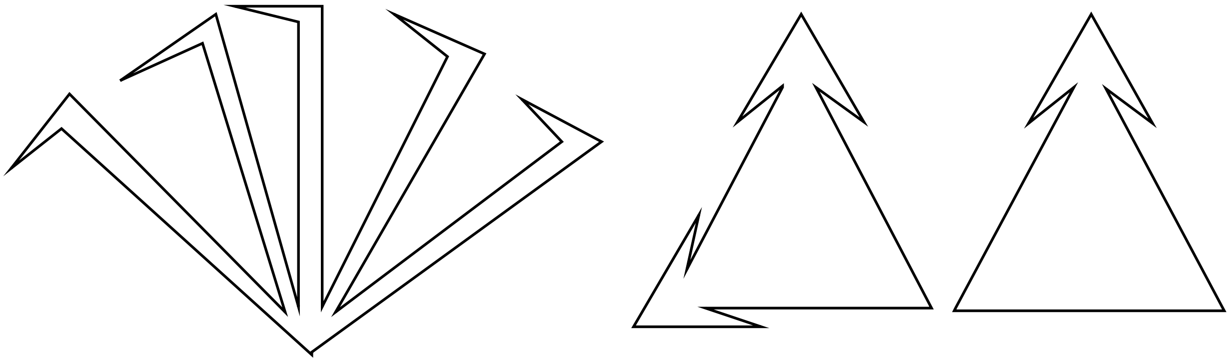


Figure 9: A polygon that requires $\lfloor \frac{n}{4} \rfloor$ edge guards and Paige and Shermer's polygons

O'Rourke's proof proceeds by induction on the number of vertices of P_n. He first establishes his result for polygons with up to nine vertices. For polygons with more than nine vertices, he considers a triangulation T of P_n. Then he shows that this triangulation contains a diagonal that cuts P_n into two polygons, P' and P'', one of which, say P', contains between five and eight edges of P. He then finds a solution for P' that can be used with any solution of P'' to obtain a solution for P_n. It is not hard to see that O'Rourke's proof of Theorem 4.9 can be implemented in linear time [97].

An interesting problem arose at this stage. A *triangulation graph* is a maximal outerplanar graph, i.e. a hamiltonean planar graph which contains n vertices and $2n-3$ edges, and all of whose internal faces are triangles. O'Rourke showed that there are triangulation graphs with n vertices such that any set of edges that covers their triangular faces requires $\lfloor \frac{2n}{7} \rfloor$ edges. Shermer later found examples of triangulation graphs that require $\lfloor \frac{3n}{10} \rfloor$ edge guards to cover them. Since this number is greater than $\lfloor \frac{n}{4} \rfloor$, this means that the technique of trying to solve Toussaint's conjecture using triangulations may not work. Recently, Shermer [109] settled the edge guarding problem for triangulated graphs. He showed:

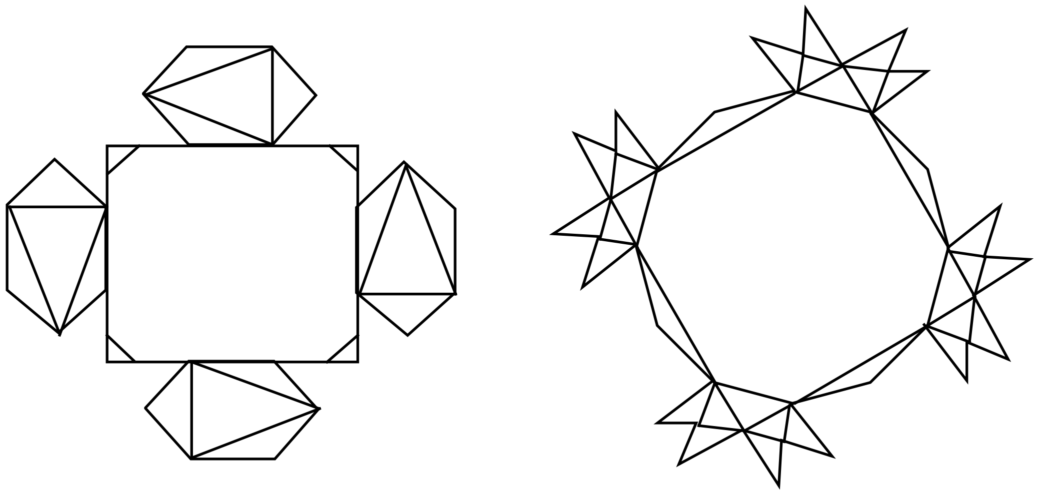
Theorem 4.10 $\lfloor \frac{3n}{10} \rfloor$ *edge guards are always sufficient and occasionally necessary to guard any triangulation graph with n vertices, with the exception of three graphs.*

The proof of Theorem 4.10 is long and tedious. For details, see Shermer's original paper. For spiral polygons, S. Viswanathan [123] showed that $\lfloor \frac{n+2}{5} \rfloor$ edge guards are always sufficient and occasionally necessary.

4.6.1 Mobile guards for orthogonal polygons

For mobile guards in orthogonal polygons, the following result by A. Aggarwal [4] gives a complete solution:

Theorem 4.11 $\lfloor \frac{3n+4}{16} \rfloor$ *mobile guards are sufficient and occasionally necessary to cover any orthogonal polygon with n vertices.*



(a) (b)

Figure 10: O'Rourke's and Shermer's polygons that require $\lfloor \frac{2n}{7} \rfloor$ and $\lfloor \frac{3n}{10} \rfloor$ edge guards respectively.

The proof of Aggarwal's result is interesting, but rather long and complicated. The interested reader can find the complete proof in [96]. It is worth noticing here that Aggarwal's proof for Theorem 4.11 leads to an $O(n^2 \log n)$ time algorithm using Sack's or Lubiw's quadrilaterization algorithms.

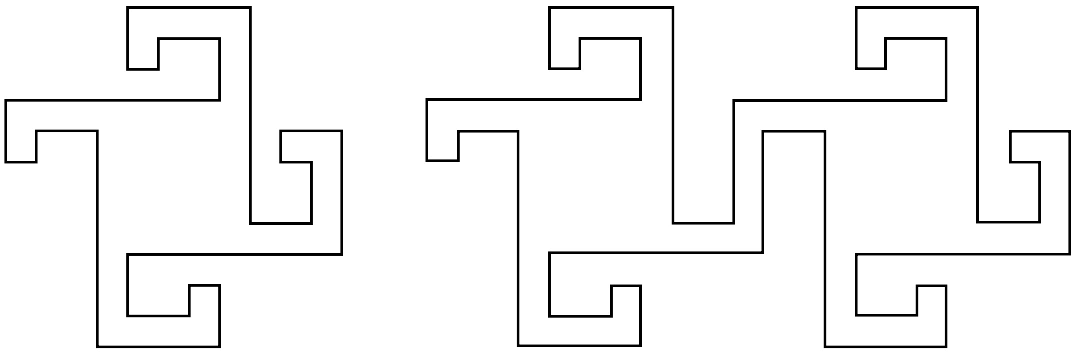


Figure 11: A polygon that requires $\lfloor \frac{3n+4}{16} \rfloor$ mobile guards.

An interesting open problem is:

Problem 4.1 *Find, if possible, a subquadratic algorithm to solve Theorem 4.11*

Aggarwal's result was generalized to orthogonal polygons with holes by Györy, Hoffmann, Kriegel and Shermer [66]. They proved:

Theorem 4.12 $\lfloor \frac{3n+4h+4}{16} \rfloor$ *mobile guards are always sufficient and occasionally necessary to guard an orthogonal polygon.*

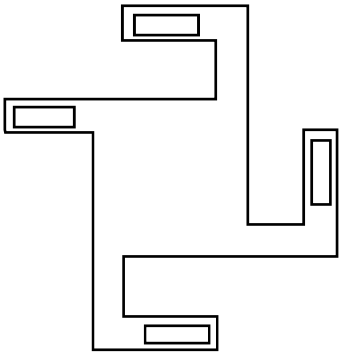


Figure 12: A polygon that requires $\lfloor \frac{3n+4h+4}{16} \rfloor$ mobile guards.

4.7 The Fortress and Prison Yard Problems

In this section, we study two variations to the Art Gallery Problem due to D. Wood:

The Fortress Problem: How many vertex guards are needed to see the exterior of a polygon?

The Prison Yard Problem: How many vertex guards are sufficient to simultaneously guard the interior and the exterior of any simple polygon with n vertices?

4.7.1 The Fortress Problem

For arbitrary polygons, the Fortress Problem was solved by O'Rourke and Wood [97]. They proved:

Theorem 4.13 $\lceil \frac{n}{2} \rceil$ *vertex guards are necessary and sufficient to to see the exterior of any polygon with n vertices.*

The upper bound of $\lceil \frac{n}{2} \rceil$ is achieved by any convex polygon with n vertices. The upper bound is proved by "triangulating" the exterior of P_n using an extra point at infinity. Next, this graph is converted to a triangulation of a polygon by splitting one of its vertices in two. This produces a graph with $n+1$ vertices that is 3-vertex colorable. If the smallest chromatic class does not contain the vertex at infinity, place a guard at the points of this chromatic class, otherwise choose the smaller of the two remaining chromatic classes. It is easy to see that this rule will not use more than $\lceil \frac{n}{2} \rceil$ guards; see [97].

For orthogonal polygons, Aggarwal [4] proved:

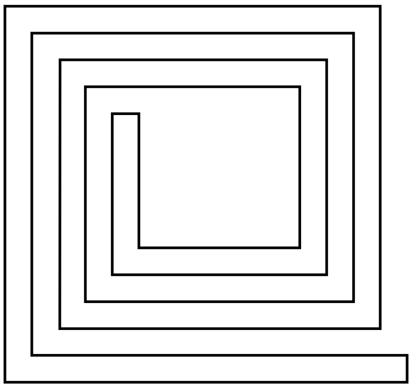


Figure 13: Aggarwal's polygon requiring $\lceil \frac{n}{4} \rceil + 1$ vertex guards.

Theorem 4.14 $\lceil \frac{n}{4} \rceil + 1$ *vertex guards are necessary and sufficient to see the exterior of any orthogonal polygon P with n vertices.*

The upper bound is proved as follows: First enclose P within a rectangle R. Then transform the area contained between R and P into an orthogonal polygon by cutting away from it a small rectangle having two of its vertices in P. From here, a small modification to the proof of Theorem 4.3 given by given by Edelsbrunner, O'Rourke and Welzl [43] gives the desired result.

The proofs of Theorems 4.13 and 4.14 lead to linear time algorithms [97].

4.7.2 Fortress problems with guards in the plane

In the previous section, the guards were required to be placed within the polygon. For the case when this condition is dropped, the following result due to Aggarwal and O'Rourke [96] solves the problem:

Theorem 4.15 $\lceil \frac{n}{3} \rceil$ *point guards are always sufficient and sometimes necessary to cover the exterior of an n vertex polygon P.*

The easiest proof for Theorem 4.15 is due to Shermer. He first adds two extra points u and v such that the convex hull of $P \cup \{u, v\}$ is a convex quadrilateral Q; see Figure 14. The area between Q and P is again triangulated and three-colored. The guarding set is then obtained from this coloring. Shermer's proof presented in [97] leads again to a linear time algorithm.

4.7.3 Edge guards and the Fortress Problem

Yiu and Choi [125] studied the Fortress Problem using edge guards. They show:

Theorem 4.16 $\lceil \frac{n}{3} \rceil$ *(respectively $\lfloor \frac{n}{4} \rfloor + 1$) edge guards are sufficient and occasionally necessary to guard the exterior of a simple (respectively, orthogonal) polygon P with n vertices.*

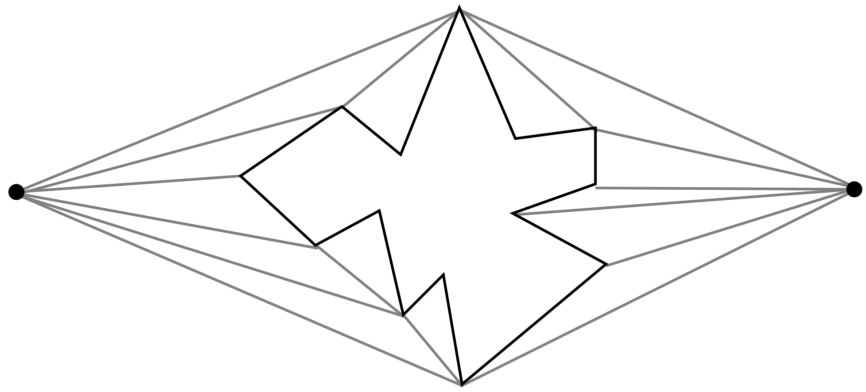


Figure 14: Illustration of proof of Theorem 4.15

For simple polygons, their proof proceeds as follows: If P is convex, it is easy to see that $\lceil \frac{n}{3} \rceil$ edge guards are necessary and sufficient. If P is not convex, and all of its pockets have at most four edges, then placing a guard at every third edge of P suffices. The remaining case is handled by a result of O'Rourke [96] that asserts that in any triangulation of a simple polygon, there is a diagonal that cuts off 4, 5, or 6 edges. Then by an analysis similar to that used to prove Theorem 4.9 our result follows. Their proof leads to a linear time algorithm.

The orthogonal part of Yiu and Choi's result follows by first enclosing the polygon within a square, then decomposing the region between this square and P into a set of $\frac{n}{2} + 2$ rectangles by extending the edges of P parallel to the x-axis until they meet another edge of P or the square. Yiu and Choi then proceed in a way similar to that used by Edelsbrunner, O'Rourke and Welzl [43] in their proof of Theorem 4.3. The result can be implemented in linear time.

Yiu [124] also studied the case when consecutive vertex guards are used. He defines a *k-consecutive vertex guard* to be a set of k vertex guards placed at k consecutive vertices on the boundary of a polygon. He shows that $\lceil \frac{n}{k+1} \rceil$ k-consecutive vertex guards are sometimes necessary and always sufficient to cover the exterior of any n-vertex polygon, $k < n$.

4.7.4 The Prison Yard Problem

For the Prison Yard Problem, the first important result was due to O'Rourke [97]. He proved that $min\{ \lceil \frac{n}{2} \rceil + r, \lfloor (n + \lceil \frac{h}{2} \rceil)/2 \rfloor, \lfloor \frac{2n}{3} \rfloor \}$ guards were always sufficient to to simultaneously guard the interior and the exterior of any polygon P with n vertices, of which r are reflex and h are in the convex hull of P. He conjectured [97] that $\lceil \frac{n}{2} \rceil$ vertex are always sufficient. This became known as the *Prison Yard Conjecture*. This problem was solved in 1992 by Füredi and Kleitman [54].

Theorem 4.17 *$\lceil \frac{n}{2} \rceil$ vertex guards (resp. $\lfloor \frac{n}{2} \rfloor$) vertex guards are always sufficient and occasionally necessary to simultaneously guard the interior and the exterior of a convex (resp. non convex) polygon with n vertices.*

The reader interested in details should consult Füredi and Kleitman's paper.

The first result for the orthogonal case was again obtained by J. O'Rourke. He proved [97] that $\lfloor \frac{7n}{16} \rfloor + 5$ vertex guards were always sufficient. This result was recently improved by Hoffmann and Kriegel [74]. They showed:

Theorem 4.18 $\lfloor \frac{5n}{12} \rfloor + 2$ *(resp.* $\lfloor \frac{n+4}{3} \rfloor$*) vertex (resp. point) guards are always sufficient to guard the interior and the exterior of an orthogonal polygon with holes.*

They also show:

Theorem 4.19 $\lfloor \frac{3n}{10} \rfloor + 2$ *(resp.* $\lfloor \frac{5n}{16} \rfloor + 2$ *) vertex guards are always sufficient and occasionally necessary to guard the interior and the exterior of a* staircase *(resp. orthoconvex) polygon with* n *vertices.*

They introduce a new problem that they call the *Prison Problem* in which the guards are required to cover all of the plane, that is, the interior, exterior and the "interior" of the holes of the orthogonal polygon. They prove:

Theorem 4.20 $\lfloor \frac{5n-4h}{12} \rfloor + 2$ *guards are always sufficient to cover all the cells, the interior and the exterior of an orthogonal polygon with holes.*

All their results are based on Theorem 4.8, and are proved in a similar way to Theorem 4.7. Hoffmann and Kriegel also show that all their results, with the exception of the orthoconvex one, lead to $O(n^2)$ time algorithms; see Theorem 4.7. The $\lfloor \frac{5n}{16} \rfloor + 2$ guards for the orthoconvex problem can be found in linear time. They also conjecture:

Conjecture 4.4 *There is a constant* c *such that any orthogonal prison yard can be watched with* $\frac{5n}{16} + c$ *vertex guards.*

We also mention the following result for *monotone* polygons. A polygon is called monotone if every line parallel to the x axis intersects it exactly once. O'Rourke [97] proved:

Theorem 4.21 $\lceil \frac{n}{2} \rceil$ *vertex guards are occasionally necessary and always sufficient to see the interior and exterior of a monotone polygon.*

5 Generalized guards and hidden sets

Let P be a polygon, and H a subset of points in P. We say that H is a *hidden set* if no pair of elements of H are visible to each other, i.e. if the line segment joining them intersects the exterior of P. The following result is due to T. Shermer [105]:

Theorem 5.1 *Computing the size of the maximum hidden set or hidden vertex set is NP-hard; computing the size of the minimum hidden guard set is NP-complete and computing the size of the minimum hidden vertex guard set is NP-hard.*

In the same paper, Shermer shows that any polygon with r reflex vertices may have a hidden set of size at most $r+1$. He also shows that that the size of the maximum vertex hidden set of a polygon with n vertices is at most $\lfloor \frac{n}{2} \rfloor$. Both bounds are tight. For orthogonal polygons, tight bounds for hidden guard and hidden vertex guard sets of $\frac{n-2}{2}$ are given.

Two points u and v of a polygon are called L_j-visible if there is a polygonal path joining them consisting of at most j straight line segments. Under this definition, two points that are visible are L_1-visible. A set Q on the plane is called L_j-convex if any two points in them are L_j-visible. Shermer [106] proved:

Theorem 5.2 $\lfloor \frac{n}{2j+1} \rfloor$ *vertex or point guards are always sufficient and occasionally necessary to guard, under L_j visibility, any polygon with n vertices.*

Theorem 5.3 *For any integers $j \geq 0$ and $n \geq j+1$, there exist polygons with n vertices that have a L_j-hidden vertex set of size $\lfloor \frac{n}{j+1} \rfloor$. Moreover, these polygons require at least $\lfloor \frac{n}{j+1} \rfloor$ regions in any covering or partition by L_j-convex regions. These bounds are tight.*

Notice that for $j = 1$, Theorem 5.2 yields Chvátal's Art Gallery Theorem. Shermer also obtained bounds to guard, under L_j visibility, the exterior of a polygon. For these and more results regarding L_j visibility, the reader is referred to [106].

Hidden sets can also be studied from the following point of view: Suppose that we have a family $F = \{S_1, \ldots, S_n\}$ of disjoint sets on the plane. A set of points H contained in the complement of the union of the elements of F is called a hidden set of F if the line segment joining any two elements of H intersects an element of F. Hurtado, Serra, and Urrutia [75] proved:

Theorem 5.4 *Let F be a family of n disjoint line segments on the plane, no two of which are collinear. Then F always has a hidden set of size at least $\sqrt{n}$. There are sets of lines that admit no hidden set of size greater than $2\sqrt{n}$.*

Their lower bound proof is based on the Erdös-Szekeres [45] theorem that asserts that any sequence of n numbers contains an increasing or decreasing subsequence of size $\sqrt{n}$. To prove their result they first shorten the segments until all have disjoint projections on the x-axis. Then they take the sequence determined by the slopes of the line segments in the order of their projection on the x-axis, and applying the Erdös-Szekeres theorem, obtain a subset S of size at least $\sqrt{n}$ of F. This set has a hidden set of size at least $|S|$ and the result follows. The upper bound is achieved by an easy example. For families of isothetic line segments, i.e. each of them parallel to the x or y-axis, they obtain a lower bound of $\lfloor \frac{n}{2} + \frac{\sqrt{n}}{2} \rfloor$ and an upper bound of $\lfloor \frac{n}{2} + \sqrt{n} \rfloor$. They can find those points in $O(n \ln n)$.

5.1 L_j visibility for orthogonal polygons

Gewali and Ntafos [61] studied *orthogonal* L_2 visibility for orthogonal polygons and *grids*, i.e. the union of a collection of horizontal or vertical line segments. In this case the polygonal line joining two L_2 orthogonally visible points has at most one horizontal line segment and at most one vertical line segment. They call this *periscope visibility*. A grid is called simple if all the endpoints of its line segments lie on the outer face of the planar subdivision induced by the grid. They proved:

Theorem 5.5 *A minimum periscope guard set for simple grids and orthogonal polygons can be found in $O(n^3)$ time.*

A T_k-guard is a tree of diameter k completely contained in a polygon P. A point is guarded by a tree T_k if it is visible from at least one point of T_k. Györy, Hoffmann, Kriegel and Shermer [66] define the function $r(n, h, k)$ to be the minimum number of T_k trees needed to guard an orthogonal polygon with n vertices and h holes. They prove the following results:

Theorem 5.6 *For k even, $r(n, h, k) \geq \lfloor \frac{n-2h}{k+4} \rfloor$. For $k = 1, 3$, $r(n, h, k) \geq \lfloor \frac{3n+(7-3k)h+4}{3k+13} \rfloor$, and for k odd, $k \geq 5$, $r(n, h, k) \geq \lfloor \frac{3(n+2h)+4}{3k+13} \rfloor$.*

Theorem 5.7 *$r(n, 0, k) \leq \lfloor \frac{n}{k+4} \rfloor$, with equality for k even; $r(n, h, 1) = \lfloor \frac{3n+4h+4}{16} \rfloor$, and $r(n, h, 2) \leq \lfloor \frac{n}{6} \rfloor$.*

The proofs of these results are based on a long case analysis. For the details, the reader is referred to the authors' paper. Some open problems posed in [66] are:

Problem 5.1 *What is the lower bound on $r(n, h, k)$ when h is large?*

Problem 5.2 *What are the exact bounds for orthogonal polygons with holes expressed as a function only of n and k?*

The authors mention that Wessel showed a lower bound of $\lfloor \frac{3n+4}{14} \rfloor$ for $k = 1$.

6 Floodlight illumination problems

In this section, we survey results concerning illumination problems using floodlights. The following problem, posed by J. Urrutia in 1992, is perhaps the most interesting open problem in this area.

Problem 6.1 The Stage Illumination Problem: *Let l be a line segment contained in the x-axis of the plane, and $F = \{f_1, \ldots, f_n\}$ be a set of floodlights with sizes $\{\alpha_1, \ldots, \alpha_n\}$ resp. such that their apexes are located at some fixed points on the plane, all on the same side of l. Is it possible to rotate the floodlights around their apexes so as to obtain a final configuration such that l is completely illuminated?*

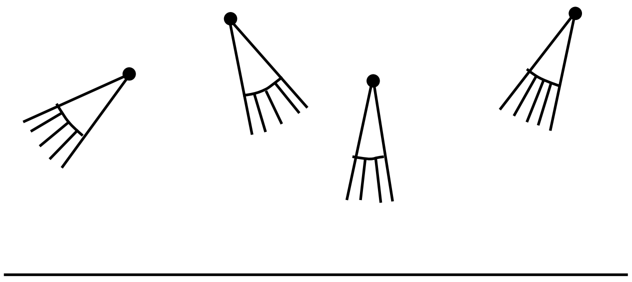


Figure 15: The Floodlight Illumination Problem.

The problem of finding an efficient algorithm to solve the Stage Illumination Problem, or proving that such an algorithm does not exist, is open. Even the cases when all the floodlights are the same size, or all lie on a line segment different from l are open.

On the other hand, the following result was proved in [12]. Suppose that we have a set $F = \{f_1, \ldots, f_n\}$ of n floodlights with angles of illumination $\{\alpha_1, \ldots, \alpha_n\}$ respectively. Let P_n be a collection of n points on the plane. We say that a floodlight f_i is assigned to a point $p_j \in P_n$ if the apex of f_i lies on p_j.

Theorem 6.1 *[12] Let P_n be a set of n points on the plane, and $F = \{f_1, \ldots, f_n\}$ a set of n floodlights with angles of illumination $\{\alpha_1, \ldots, \alpha_n\}$ resp. such that $\alpha_1 + \ldots + \alpha_n = \pi$. Then it is always possible to assign one floodlight to each point of P_n and position the floodlights in such a way that the whole plane is illuminated.*

In the next figure, we show a solution to the plane illumination problem for the case of four points and four $\frac{\pi}{2}$-floodlights.

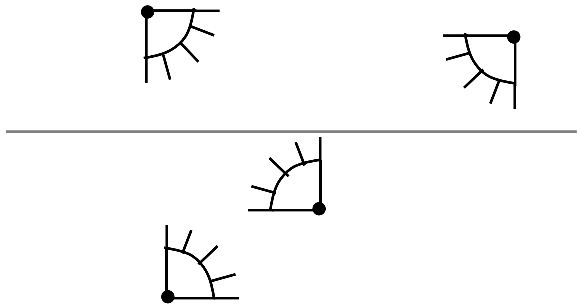


Figure 16: Four $\frac{\pi}{2}$-floodlights that illuminate the plane.

The main step in the proof of Theorem 6.1 is to split the floodlights into three disjoint sets $\{f_1, \ldots, f_i\}$, $\{f_{i+1}, \ldots, f_k\}$, and $\{f_{k+1}, \ldots, f_n\}$ such that $\alpha_1 + \ldots + \alpha_i = \beta_1$, $\alpha_{i+1} + \ldots + \alpha_j = \beta_2$ and $\alpha_{j+1} + \ldots + \alpha_n = \beta_3$ such that β_1, β_2 and β_3 are all less than or equal to π. The plane is then subdivided into three unbounded angular "wedges"

w_i with apertures β_i, $i = 1, \ldots, 3$ such that w_1, w_2 and w_3 contain exactly i, $j - i$ and $n - j$ elements of P_n respectively. Then they solve each of these "wedge illumination problems" separately to obtain a solution to the plane illumination problem. Their solution leads to an $O(n \log n)$ time algorithm. For a time, it was not known if the complexity of the previous problem, called the *Floodlight Illumination Problem of the Plane*, was optimal or not.

Motivated by the proof of Theorem 6.1, B. Steiger and I. Streinu showed that the partitioning of the plane into w_1, w_2 and w_3 can be achieved in linear time; see [111, 112, 110]. Moreover, they showed that the *tight floodlight wedge illumination problem* as defined before is in NP. Moreover, they proved [111]:

Theorem 6.2 *The Floodlight Illumination Problem of the Plane has a lower bound complexity of* $\Omega(n \log n)$.

This result has been generalized to higher dimensions by G. Rote [100] in the following way: Let P be a convex polyhedron in R^d, and p a point in the interior of P. Each $d-1$ facet c_i of P, together with p, defines a cone of illumination f_i, consisting of all semirays starting at p that intersect c_i. Let $F(P,p) = \{f_i : c_i \textit{ is a } d-1 \textit{ facet of } P\}$.

Theorem 6.3 *[100] Let P be a convex polyhedron in R^d with m $d-1$ facets, P_m a set of m points in R^d and let $F(P,p)$ be as defined above. Then it is always possible to assign one floodlight of $F(P,p)$ to each point of P_m and position them in such a way that R^d is completely illuminated.*

Surprisingly, the following variation of the Stage Illumination Problem can be solved efficiently.

Optimal Floodlight Illumination of Stages: Let l be a line segment contained in the x-axis, and P_n a set of points on the plane with positive y-coordinate. Find a set of floodlights $F = \{f_1, \ldots, f_m\}$ such that the apex of every f_i is located at an element of P_n, F illuminates l, and $\alpha_1 + \ldots + \alpha_m$ is minimized.

In the previous problem, we allow more than one floodlight to have its apex at the same point. Czyzowicz, Rivera-Campo and Urrutia [39] proved:

Theorem 6.4 *The Optimal Floodlight Illumination of Stages problem can be solved in* $O(n \log n)$ *time.*

We notice that the solutions obtained in [39] for Theorem 6.4 require two floodlights in *at most* one element of P_n. If we insist that at each point we have exactly one floodlight, the problem remains open.

In [48], the *Two Floodlight Illumination Problem* for convex polygons is studied. In this problem, we want to illuminate a convex polygon P using at most two floodlights in such a way that the sum of their sizes is minimized. In [48] the following result is proved:

Theorem 6.5 *The Two Floodlight Illumination Problem for convex polygons with n vertices can be solved in $O(n^2)$ time.*

It is not known if the algorithm to solve the previous problem is optimal. The k-floodlight illumination problem, that is the problem of illuminating a convex polygon P with k floodlights, $k \geq 3$, such that the sum of their sizes is minimized, is open. At this point we do not even know if the floodlights have to be located at vertices of the polygon to be illuminated. Moreover, we do not even know if there is a constant l such that if $k > l$ then the optimal solution to the k-floodlight illumination problem uses at most l floodlights. We know, however, that if the vertices of P are cocircular, two floodlights suffice.

We close this section with a result of O'Rourke, Streinu and Shermer related to the Three-Floodlight Illumination problem presented in Section 3.4. As we showed in Theorem 3.1, any convex polygon can always be illuminated with three vertex floodlights such that the sum of their sizes is π. O'Rourke, Streinu and Shermer [99] showed that in general, Theorem 3.1 cannot be extended. In a very nice paper they prove:

Theorem 6.6 *Let $\{f_1, \ldots, f_m\}$ be a set of m floodlights of sizes $\{\alpha_1, \ldots, \alpha_m\}$ such that $\alpha_1 + \ldots + \alpha_m = \pi$, m large enough and P a convex polygon with n vertices, $n \geq m$. Then it is not always possible to illuminate P by assigning at most one floodlight to each vertex of P.*

An open problem is that of determining the smallest integer for which Theorem 3.1 can be generalized. We do not know if Theorem 3.1 can be extended, even for the case $n = 4$; see [99].

6.1 Floodlight illumination of orthogonal polygons

For orthogonal polygons, the first result was obtained by Estivill-Castro and Urrutia [48]. A floodlight is called *orthogonal* if it is of size $\frac{\pi}{2}$.

Theorem 6.7 *Any orthogonal polygon with n vertices can always be illuminated using at most $\lfloor \frac{3n-4}{8} \rfloor$ orthogonal vertex floodlights. If the floodlights are allowed to be anywhere on the boundary of the polygon, $\lfloor \frac{n}{4} \rfloor$ suffice. Both bounds are tight.*

The proof for the $\lfloor \frac{3n-4}{8} \rfloor$ vertex guards was proved by obtaining four different illumination rules as follows: Classify the edges of an orthogonal polygon P into four types, *top*, $left$, *bottom* and $right$ edges. The *left-top* illumination rule is now defined as follows: At the top vertex of all $left$ edges, and at the left endvertex of all top edges, put an orthogonal floodlight that illuminates the angular sector $\frac{3\pi}{2}$ to 2π. It is easy to see that these floodlights illuminate P. In a similar way, we can define the *top-right*, *right-bottom* and *bottom-left* illumination rule. An easy counting argument shows that one of these illumination rules uses at most $\lfloor \frac{3n-4}{8} \rfloor$ floodlights. The examples in Figure 17 achieve this bound.

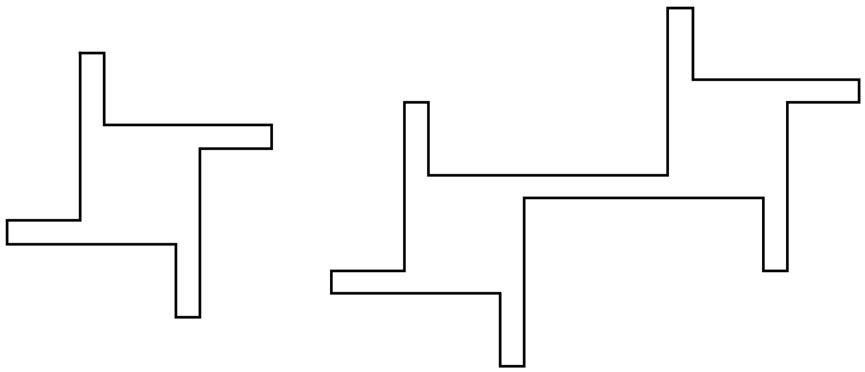


Figure 17: Orthogonal polygons that need $\lfloor \frac{3n-4}{8} \rfloor$ orthogonal floodlights

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The proof for the $\lfloor \frac{n}{4} \rfloor$ bound follows an idea similar to that used by Edelsbrunner, O'Rourke and Welzl [43] in the proof of Theorem 4.3. It is interesting to point out here that the bound of $\lfloor \frac{n}{4} \rfloor$ obtained in Theorem 6.7 is the same as that of Theorem 4.3 for orthogonal polygons using vertex guards. Thus Theorem 6.7 provides a new proof for Theorem 4.3.

This result was later extended to orthogonal polygons with holes by Abello, Estivill-Castro, Shermer and Urrutia [1]. They showed:

Theorem 6.8 $\lfloor \frac{3n+4(h-1)}{8} \rfloor$ *orthogonal vertex floodlights are always sufficient and occasionally necessary to illuminate any orthogonal polygon with n vertices and h holes. The bound is tight.*

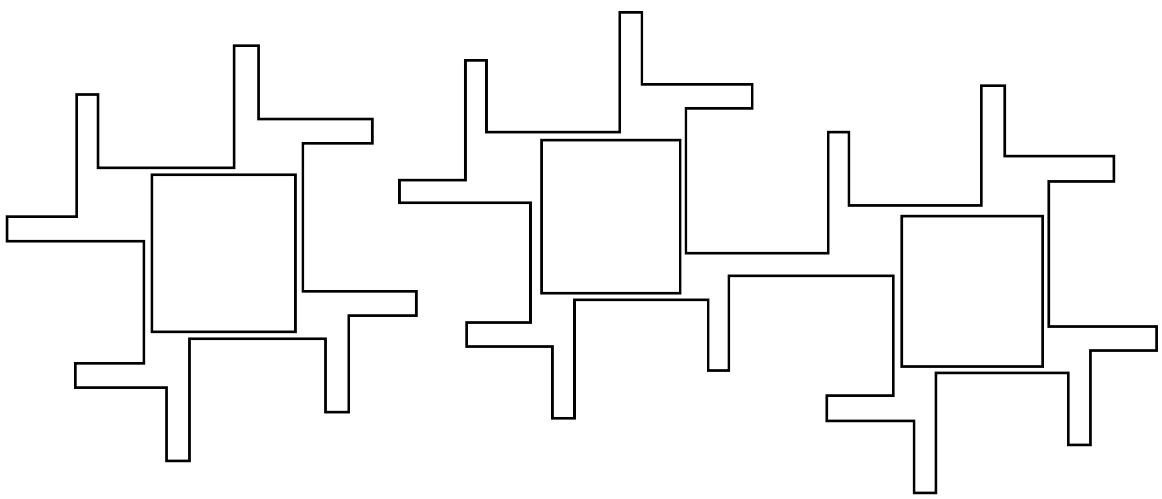


Figure 18: An orthogonal polygon with holes that needs $\lfloor \frac{3n+4(h-1)}{8} \rfloor$ orthogonal floodlights.

Theorem 6.8 gives the first tight bound for vertex guards of orthogonal polygons with holes.

6.2 π-floodlights

The results of Theorem 6.7 motivated the study of the following problem, posed by Urrutia. Is there an $\alpha < \pi$ such that any polygon P can be illuminated by placing an α-floodlight at every vertex of P? It is easy to see that $\alpha = \pi$ is sufficient. Take any triangulation of P, choose an ear e of it and place a π-floodlight at the vertex of degree two of e. Cut e off from P and proceed by induction. The original value conjectured by Urrutia was $\alpha = \frac{\pi}{2}$. O'Rourke and Xu [98] disproved this conjecture. The reader can easily verify that the polygon shown in Figure 19 cannot be illuminated by placing a $\frac{\pi}{2}$-floodlight at each of its vertices. Shortly after their result was proved, it was extended by Estivill-Castro, O'Rourke, Urrutia and Xu [46]. They proved the following result:

Theorem 6.9 *For any $\alpha < \pi$ there is a polygon P that cannot be illuminated by placing an α floodlight at every vertex of P.*

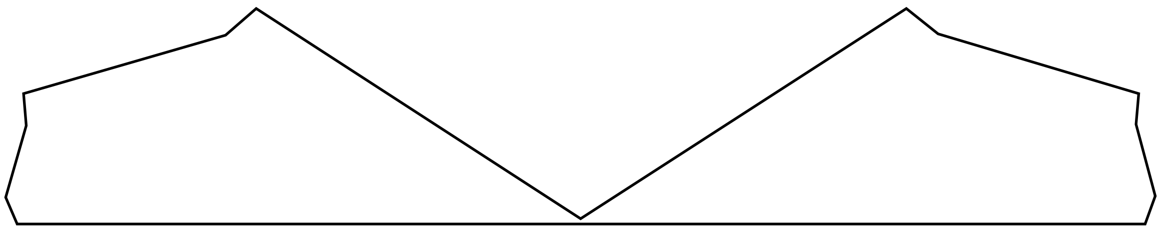


Figure 19: O'Rourke and Xu's polygon cannot be illuminated by a $\frac{\pi}{2}$-floodlight at each vertex.

It is now natural to ask the following open question:

Problem 6.2 *How many π vertex floodlights are always sufficient to illuminate any polygon with n vertices? Is there a $c < 1$ such that any polygon with n vertices can be illuminated with cn vertex floodlights?*

F. Santos produced a family of polygons with $5n + 1$ vertices that require $3n$ π-vertex floodlights. The only known upper bound at this time is $n-2$ which is obviously not tight.

For point floodlights, Bunting and Larmann [15], and independently Csizmadia and Tóth, proved in 1992 that $\lfloor \frac{4}{9}(n + \frac{1}{4}) \rfloor$ π-floodlights suffice. In a recent paper, Csizmadia and Tóth proved [31]:

Theorem 6.10 $\lceil \frac{2}{5}(n - 3) \rceil$ *point π-floodlights are always sufficient to illuminate any polygon P with n vertices, $n > 3$.*

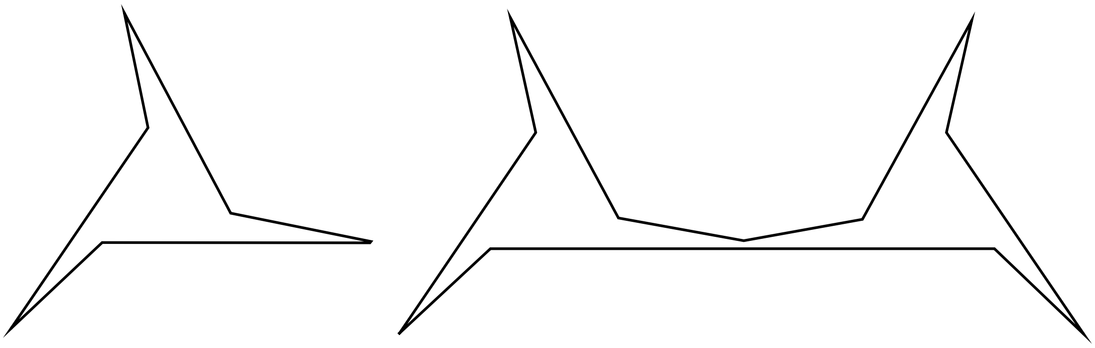


Figure 20: Santos' polygons with $5n+1$ vertices, which require $3n$ π-vertex floodlights.

To prove Theorem 6.10, Csizmadia and Tóth first find a triangulation T of P. Then using the dual tree of T, they show that they can cut P along a diagonal of T chosen appropriately. They then proceed by induction on the number of vertices of P.

From Theorem 6.10, it follows that $\lceil \frac{\pi}{\alpha} \rceil \lceil \frac{2}{5}(n-3) \rceil$ α-point floodlights are always sufficient to illuminate an n-vertex polygon. This bound is not optimal; for example it is easy to see that $n-2$ $\frac{\pi}{3}$-point floodlights always suffice to illuminate any polygon with n vertices, and for $\alpha = \frac{\pi}{3}$, the previous formula yields a value greater than $n-2$. The proof of Theorem 6.10 leads to a linear time algorithm. We close this section with the following conjecture:

Conjecture 6.1 $\lfloor \frac{n}{3} \rfloor$ *point π-floodlights are always sufficient to illuminate any polygon with n vertices.*

Urrutia [121] wrote a survey (in Spanish) containing many results on floodlight illumination problems.

7 Illuminating families of convex sets on the plane

As we mentioned in Section 3.3, Fejes Tóth showed in 1977 that any family of n disjoint compact convex sets can always be illuminated with at most $4n-7$ light sources, $n \geq 3$. This result was rediscovered in 1989 by Zaks and Urrutia [122]. Fejes Tóth also proved:

Theorem 7.1 $2n-2$ *light sources are always sufficient and occasionally necessary to illuminate any family of n disjoint closed circles.*

This result was rediscovered again by Coullard *et al* [28] and Czyzowicz *et al* [35]. Czyzowicz, Rivera-Campo, and Urrutia [34] also studied the problem of illuminating families of triangles and rectangles. They proved that any family of n disjoint triangles can always be illuminated with at most $\lfloor \frac{4n+4}{3} \rfloor$ guards. They also showed:

Theorem 7.2 $n+1$ *lights are always sufficient and $n-1$ occasionally necessary to illuminate any family of n homothetic triangles.*

The example shown in [34] can be slightly modified to obtain a lower bound of n lights, instead of $n-1$ for Theorem 7.2; see Figure 21.

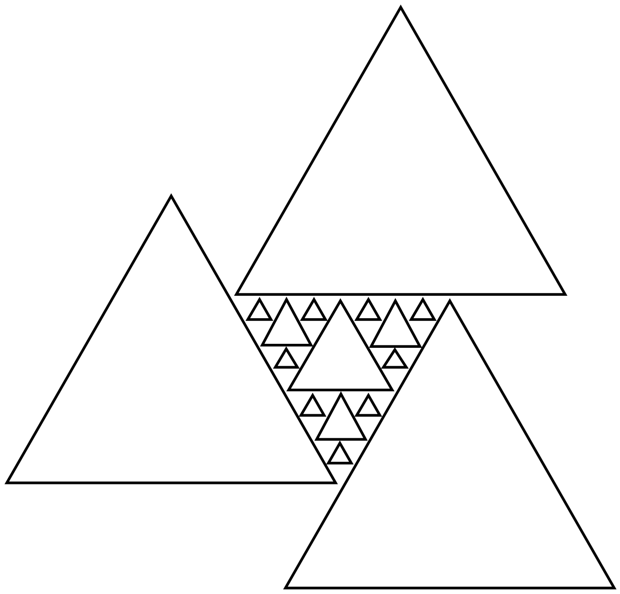


Figure 21: A family of n homothetic triangles that requires n point lights.

We now prove:

Theorem 7.3 *Any family F of n disjoint k-gons can always be illuminated with at most $\lfloor \frac{nk+n+3}{3} \rfloor$ lights.*

To prove this result, we simply enclose all the elements of F within a triangle. This can now be considered as a polygon with $kn+3$ vertices with n holes. Thus by Theorem 4.5 the elements of F can be illuminated with at most $\lfloor \frac{nk+n+3}{3} \rfloor$ lights. Our result follows.

For $k=3$, this slightly improves on Czyzowicz, Rivera-Campo, and Urrutia's [34] result on triangles.

There are still some small gaps left between the upper and lower bounds to illuminate families of rectangles and triangles. The lower bounds are:

1. $n-1$ to illuminate families of n isothetic rectangles

2. n to illuminate any family of homothetic triangles

An open problem is that of closing these gaps, and of establishing non-trivial lower bounds for the general case of triangles. Furthermore, Theorem 7.3 provides an upper bound for the problem of illuminating families of convex k-gons, $k \geq 3$. None of these bounds, I believe, are tight. It would be of interest to tighten them, particularly for the case when $k \leq 12$ where better bounds are obtained using Fejes Tóth's Theorem 3.3. For the case when our sets are arbitrary disjoint triangles, we conjecture:

Conjecture 7.1 *[34] There is a constant c such that $n + c$ point lights are sufficient to illuminate any collection of n triangles*

For families of disjoint isothetic rectangles Czyzowicz, Rivera-Campo, and Urrutia [34] proved that $\lfloor \frac{4n+4}{3} \rfloor$ guards suffice, and conjectured that $n + c$ would suffice, c a constant. This conjecture can easily be obtained from Theorem 4.6. We now prove:

Theorem 7.4 *Any family F of n disjoint rectangles can be illuminated with at most $n + 1$ light sources.*

To prove our result, we enclose the elements of F by a big rectangle, and consider this as an orthogonal polygon with holes. The total number of vertices is now $4n + 4$. By Theorem 4.6 this can be illuminated with $n + 1$ light sources.

Everett and Toussaint [51] proved that any family of n disjoint squares, $n > 4$, can always be illuminated with n point lights. Everett, Lyons, Reed and Souvaine [49] also showed that if the squares are such that they all intersect a line L, then $\lfloor \frac{2n}{3} \rfloor + 2$ lights are sometimes required and $\lfloor \frac{2n}{3} \rfloor + 7$ are always sufficient. They also gave $O(n \ln n)$ time algorithms to place the n, and $\lfloor \frac{2n}{3} \rfloor + 7$ lights respectively.

García-López [55] studied the problem of illuminating the free space generated by a family of disjoint polygons, i.e. the complement of their union. Using Theorem 4.5 he showed:

Theorem 7.5 *The free space generated by any family of h disjoint polygons with a total of n vertices can be illuminated with at most $\lfloor \frac{n+h+3}{3} \rfloor$ point lights. There are families that require $\lfloor \frac{n+h-1}{3} \rfloor$ lights.*

Using Theorem 4.5 this leads to an $O(n^2)$ time algorithm. For vertex lights García-López proved:

Theorem 7.6 *$\lfloor \frac{5n}{9} \rfloor$ vertex lights are always sufficient and $\lceil \frac{n}{2} \rceil$ occasionally necessary to illuminate the free space generated by a family of disjoint polygons with n vertices.*

The proof of Theorem 7.6 leads to an $O(n \ln n)$ time algorithm to find the $\lfloor \frac{5n}{9} \rfloor$ vertex lights.

He also proved that to illuminate the free space generated by any family of m disjoint quadrilaterals, $2m$ vertex lights are always necessary and occasionally sufficient, and that $\lfloor \frac{5m+3}{3} \rfloor$ point guards are always sufficient. He conjectured that the free space generated by m disjoint quadrilaterals can always be illuminated by $n + c$ point lights, c a constant.

This conjecture was proved to be false by Czyzowicz and Urrutia [41]. They produced a family of $n = 3m - 3$ quadrilaterals such that to illuminate the free space they generate requires $4m - 4$ point lights, $m \geq 4$; see Figure 22.

The illumination problem for higher dimensions is completely different. In [38] it is shown that there are families of $O(n^2)$ boxes such that to illuminate the free space generated by them requires $O(n^3)$ light sources. On the other hand, Czyzowicz, Gaujal, Rivera-Campo, Urrutia and Zaks [40] proved:

Theorem 7.7 *For any compact convex set T in E^d, there is a constant $c_d(T)$ such that every family F consisting of n mutually disjoint congruent copies of T can be illuminated with $c_d(T)\, n$ lights.*

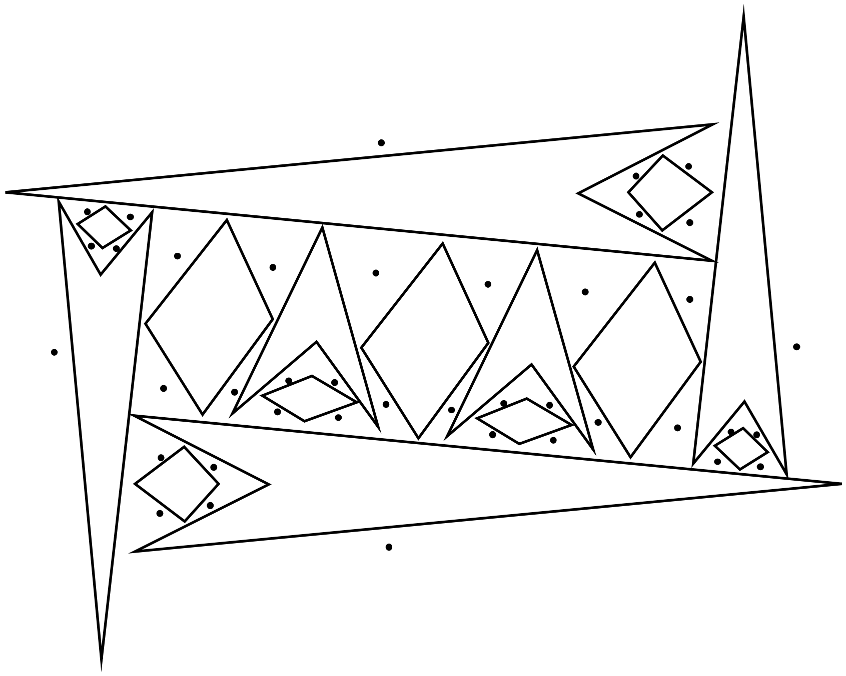


Figure 22: An example of family of with $n = 3m-3$ quadrilaterals that requires $4m-4$ point lights to illuminate the free space they generate, $m = 6$.

7.1 Illuminating line segments on the plane

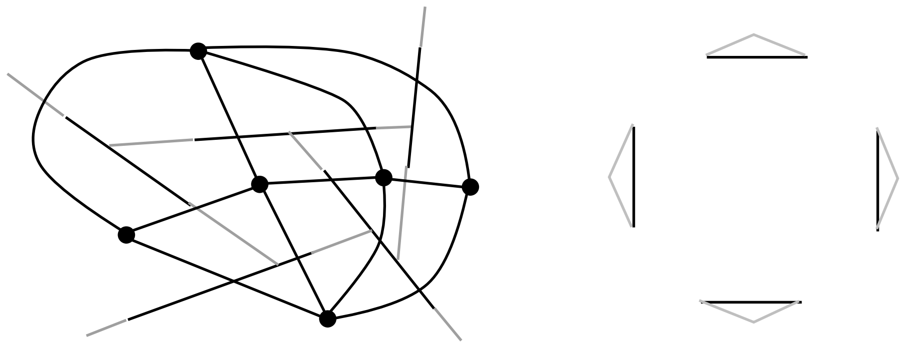
A particularly interesting problem is that of illuminating the plane in the presence of obstacles represented by collections of disjoint line segments, or illuminating the line segments themselves. J. O'Rourke [96] proved:

Theorem 7.8 $\lfloor \frac{2n}{3} \rfloor$ *point guards are sometimes necessary and always sufficient to cover the plane in the presence of n disjoint line segment obstacles, where the guards may be stationed on top of an obstacle, $n \geq 5$.*

Here is a brief outline of O'Rourke's proof: Let F be a collection of n disjoint line segments, no two of which are parallel. One by one, extend the line segments until they hit another line, or extension of another line of F. This produces a partitioning P of the plane into $n+1$ convex sets with disjoint interiors. Take the dual graph G of P; see Figure 23(a). It is now easy to see that G satisfies the conditions of Nishiseki's Theorem 2.2, and thus has a matching of size at least $\lceil \frac{n+6}{3} \rceil$. If two regions are matched, we guard them with the same guard; to each unmatched region, we assign a different guard. It is easy to see that this rule uses at most $\lfloor \frac{2n}{3} \rfloor$ point guards. The case when parallel lines appear can be handed in a similar way.

If the guards are required to be at endpoints, then Shermer and Boenke showed that n guards are sometimes needed and always sufficient [96].

The problem of illuminating families of line segments leads to an interesting variation; the illumination of a point of a line segment l can be achieved by locating a light



(a) (b)

Figure 23: (a) A family of line segments, their extensions and their dual graph. (b) An example where n vertex guards are needed to illuminate the plane.

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source at either side of l. The first result in this direction was obtained by Czyzowicz, Rival and Urrutia [33]. They proved that any collection of n line segments can always be illuminated using at most $\lfloor \frac{3n}{4} \rfloor$ line segments. This result was improved by Czyzowicz, Rivera-Campo, Urrutia and Zaks [37]. They proved:

Theorem 7.9 $\lceil \frac{2n}{3} \rceil - 3$ *point guards are always sufficient to illuminate any family F of n line segments. If the elements of F are all parallel to the x- or y-axis, $\lceil \frac{n+1}{2} \rceil$ suffice.*

The proofs of these bounds are obtained by generating planar graphs which satisfy Nishizeki's and Tutte's theorems for matchings in planar graphs. They follow similar steps to those used by O'Rourke in the proof of Theorem 7.8.

Another line illumination problem was studied by Jennings and Lenhart [77]. They were concerned with the following problem: Given a family F of n disjoint line segments, find a subset S of F such that every element of F is visible from at least one point in an element of S. They proved:

Theorem 7.10 *Given any set F of n pairwise disjoint line segments, it is possible to find a subset $S \in F$ with at most $\lfloor \frac{n}{2} \rfloor$ elements such that every element of F is seen by at least one point on one of the elements of S. The bound is tight.*

7.2 Illuminating using line segments

We now consider the problem of illuminating families of disjoint compact convex sets using line segment illuminators. Our line segments are not allowed to intersect any of the compact sets. Using the same idea used to prove Fejes Tóth's Theorem 3.3 we can prove:

Theorem 7.11 *Any family of n disjoint compact convex sets can always be illuminated with at most $n-1$ line segments, $n > 2$.*

Consider a family $F = \{S_1, \ldots, S_n\}$ of n disjoint compact convex sets. Find a family of sets $T = \{T_1, \ldots, T_n\}$ as in the proof of Theorem 3.3. Let H be the complement of $T_1 \cup \ldots \cup T_n$. Again the worst case arises when H is partitioned into triangular faces. We now construct a graph G whose vertex set is the faces of H, two of which are adjacent if they have two common elements of T on their boundaries; see Figure 24. Notice that G has $2n-4$ vertices. The degree of each vertex of G is 3, and it is 3-connected, $n > 2$. Therefore by a well known result in graph theory, G has a perfect matching. Observe that if two triangular faces of H are adjacent in G, they can be illuminated by a common line segment not intersecting the *interior* of any T_i, $i = 1, \ldots, n$. Using the perfect matching of G, we can now choose $n-2$ line segments to illuminate all the faces of H, except the outer one, which needs an extra line segment, i.e. we use $n-1$ segments in total. These line segments also illuminate all the elements of S, and our result follows.

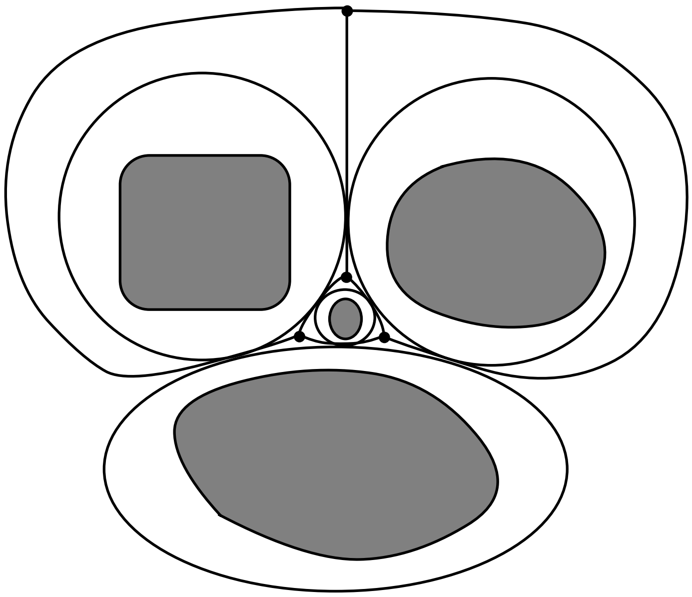

Figure 24: Illustration of proof of theorem on illuminating disjoint compact convex sets.

It is easy to see that the family of circles shown in Figure 32 (see page 42) requires $n-1$ line segment illuminators.

We close this section with the following conjecture:

Conjecture 7.2 *[33] Any family of disjoint line segments can always be illuminated with at most $\frac{n}{2} + c$ light sources, c a constant.*

8 Watchman, Robber, Safari, and Zoo-keeper's Routes

8.1 The Watchman Route Problem

The following guarding problem, known as *the Watchman Route*, was introduced by Ntafos and Chin [22]. Suppose a guard has to patrol a polygon P. To do this, he must find a closed walk W starting and ending at a *starting point* s such that every point in P is visible from some point in W; see Figure 25. In order to minimize the distance the guard has to travel, we would like to find the shortest possible route the guard can use. The following result was proved by Ntafos and Chin [22]:

Theorem 8.1 *The Optimum Watchman Route for orthogonal polygons can be solved in linear time. For polygons with holes the problem is NP-hard, even if the holes are convex or orthogonal.*

The NP-hard part was proved by reducing the Geometric Traveling Salesman Problem to the Watchman Problem. The NP-hardness for orthogonal case follows from the observation that the Geometric Traveling Salesman Problem remains intractable even under the Manhattan distance. To find the Watchman Route in orthogonal polygons, Ntafos and Chin first identify edges that have to be present in the route, then "unroll" the remaining polygon by reflecting it around several of its edges.

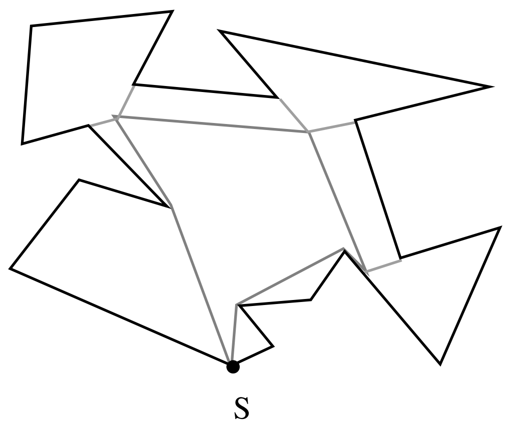

Figure 25: A watchman route.

In another paper, Chin and Ntafos [23] proved that the Watchman Problem for simple polygons can be solved in polynomial time. They showed:

Theorem 8.2 *The Watchman Route Problem for simple polygons can be solved in* $O(n^4)$ *time.*

Using techniques similar to those of Chin and Ntafos, Tan, Hirata and Inagaki [116] obtained an $O(n^3)$ algorithm to solve the Watchman Route Problem. Finally, using a divide and conquer approach, the same authors [117] showed:

Theorem 8.3 *The Watchman Route Problem can be solved in $O(n^2)$ time.*

The Watchman Route Problem *without specifying a starting point* remains open. One of the most interesting problems here, due to Chin and Ntafos [23], is:

Problem 8.1 *Can the Watchman Route Problem without an initial starting point be solved in polynomial time?*

The *External* Watchman Route Problem is that of finding a watchman route in which a watchman is required to patrol the *exterior* of a polygon. Ntafos and Gewali [94] gave an $O(n^4)$ time algorithm for finding a shortest external watchman route for a simple polygon. In the same paper, linear time algorithms to solve the Watchman Route Problem for monotone, convex, or rectilinear polygons are given. Nilsson and Wood [90] obtained a linear time algorithm to find a shortest watchman route, internal as well as external, in spiral polygons. In the same paper, they also studied the problem of finding optimal routes for m watchmen in a spiral polygon. They give an $O(n^2m)$ algorithm for this case.

Gewali and Lombardo [59] studied the problem of finding the optimal watchman route to guard the exterior of two convex polygons such that their total number of vertices is n. They gave an $O(n^3)$ algorithm to solve this problem. This was recently improved by Gewali and Ntafos [62] to $O(n^2)$. It is believed that an $O(n \ln n)$ algorithm may exist to solve this problem. The shortest external watchman route for two convex polygons need not be in contact with their boundary; this is the main source of difficulty for finding the shortest route; see Figure 26.

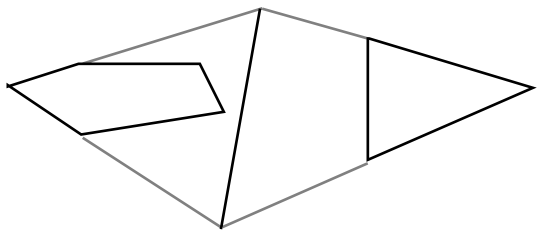

Figure 26: An external watchman route that does not intersect the boundary of two convex polygons.

An interesting result was obtained recently by Mata and Mitchell [88] concerning approximation algorithms to find watchman routes in polygons in orthogonal polygons with holes. They showed:

Theorem 8.4 *There is an $O(n^9)$ time algorithm that finds a watchman route in orthogonal polygons whose length is within $O(\log m)$ times of the optimal watchman route, where m is the minimum number of edges in a rectilinear optimal watchman route.*

The shortest watchman route in which the starting position is not specified is much harder to solve. It is generally believed that this problem can be solved in polynomial time for simple polygons, however to our knowledge, no efficient algorithm to solve this problem is known. In a recent paper, Kumar and Madhavan [81] give an algorithm to solve the watchman route without a starting point for polygons that contain an edge e such that every point of P is visible from some point in e. Their algorithm works in $O(n^{10})$.

8.2 The Robber Route Problem

Another variation introduced by Ntafos [92] is called *The Robber Route Problem.* Consider a set T of points and a set S of edges of a polygon P. Our goal is to find a closed walk W starting and ending at a point x such that every point on an element of S is visible from a point in W while W is not visible from any point in T. One could think of the elements of T as threats, and the elements of S as sights one may wish to see. If T has at most n elements, Ntafos proves [92]:

Theorem 8.5 *The Robber Route Problem can be solved in $O(n^4)$ time for arbitrary polygons, and in $O(|S|n + |T|n + n)$ time in orthogonal polygons.*

A problem closely related to the Robber Route Problem, introduced by Gewali, Meng, Mitchell, and Ntafos [60], is the computation of watchman routes inside a polygon with threat sources. The objective is to determine a watchman route that minimizes total exposure to threats; such a route is called the *least-risk* watchman route. It has been shown that a least-risk watchman route through a specified point x on the boundary of a simple orthogonal polygon with k threats can be constructed in $O(k^2n^3)$ time [60].

8.3 The Zoo-Keeper's and the Safari Route Problems

Chin and Ntafos also considered two variations to the watchman route that they called *The Zoo-keeper* and *Safari Routes*. In these problems, our objective is to walk within a polygon P containing k "sites" one wishes to visit. These sites are represented by polygons within P. In the *Safari Route* Problem we are allowed to enter the sites, as one does when visiting pavilions at an exhibition. In the *Zoo-Keeper Routes*, we are not allowed to enter the sites, as a zoo-keeper who has to feed the animals, but does not enter their cages. They showed that in general, these problems are NP-hard. However if the sites are attached to the boundary of P, efficient solutions can be obtained; under this condition, Chin and Ntafos [24] proved:

Theorem 8.6 *The Zoo-keeper's Problem can be solved in $O(n^2)$ time.*

This result has been improved by Hershberger and Snoeyink [70] who obtained an $O(n\,log^2 n)$ algorithm to solve this problem.

Ntafos has also been able to obtain a polynomial time algorithm to solve the Safari Route Problem, again under the restriction that the sites are attached to the boundary of P, in [93]. He shows:

Theorem 8.7 *The Safari Route Problem can be solved in $O(n^3)$ time.*

In the same paper, Ntafos studies watchman routes under *limited visibility*. In this context, he defines points to be d-visible if they are visible and at distance at most d from each other. In [93] Ntafos gives approximation algorithms for the d-Watchman Problem, i.e. finding a closed path that covers all of the boundary of P, and the d-sweeper problem, i.e. a path that covers the whole polygon.

An interesting case arises when the sites to be visited are exactly all the edges of a polygon P. Czyzowicz *et al* gave an $O(n)$ time algorithm to solve this case. They call this problem *The Aquarium Keeper's Problem* [32].

8.4 Vision points on watchman routes

Carlsson, Nilsson and Ntafos [17] studied another variation of the Watchman Route Problem. In this variation, the guard is restricted to move along a path W such that all of P is visible from W. However, the guard surveys the polygon around him only at some selected points of W from which he can cover all of P. They call these points *vision points*. The motivation here is that we could think of a watchman route as an electrical wire that covers all of P. The vision points are the locations along the wire where we need to install lamps to illuminate P; see Figure 27. Their problem is that of determining the minimum number of vision points or lamps needed to illuminate P. They show:

Theorem 8.8 *Finding the minimum number of vision points along a shortest watchman route is NP-hard.*

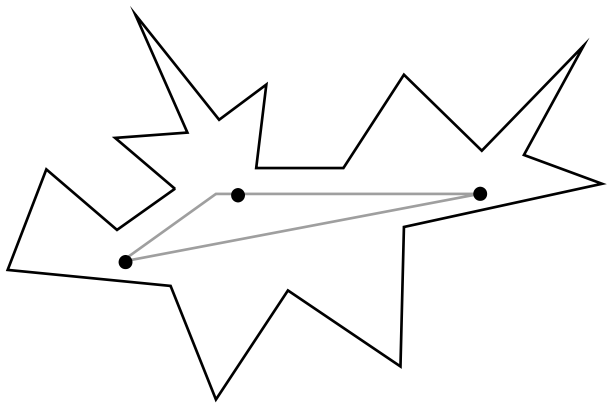

Figure 27: An example where 3 vision points are required.

For histograms, however, they can solve this problem in linear time. In the same paper, they also solve the problem of finding the best m-watchmen routes for histograms. Their algorithm runs in $O(n^3 + n^2m^2)$ time using $O(n^2m^2)$ space.

More results on watchman routes can be found in Nilsson's Ph.D. Thesis [89]. He studies watchmen routes for spiral, doughnut, histogram, and Alp polygons. For example, he shows that the number of vision points required to cover a polygon can be arbitrarily large, and if no general position is assumed, an infinite number of vision points may be required. He also shows that finding m minsum-watchmen routes in Alp polygons can be done in $O(n(n+m^3))$ time using $O(nm^3)$ space.

Very few results dealing with parallel algorithms for Watchman Route Problems have been obtained. Gewali and Stojmenovic [63] have shown that a shortest external watchman route for a convex polygon can be computed in $O(\log n)$ time using $O(\frac{n}{\log n})$ processors in the CREW-PRAM computational model. In the same paper it is proved that a shortest external watchman route for a convex polygon can be computed in $O(\log n)$ time on a hypercube with $O(n)$ processors.

9 Mobile guards

In this section we study several problems similar to Watchman Route Problems. In contrast to staionary guards, which are not allowed to move, in the following set of problems, guards are required to move. This drastically reduces the number of guards required to patrol a polygon.

9.1 The Hunter's Problem

Suzuki and Yamashita [115] considered a search problem in which both a guard and a robber are allowed to move freely within a polygon. The guard moves with bounded velocity, and the robber moves with unbounded velocity. They study variations in which the guard can see in all directions at once, or in a fixed number k of directions, which, as in the searchlight problem, can change continuously. They call such a guard a *k-searcher*. They show that there are polygons which are 2-searchable but not 1-searchable. They also give some necessary and some sufficient conditions for a polygon to be searchable. For example they prove that if a polygon is 1-searchable, it has no points x_1, x_2, and x_3 such that the shortest path between x_i and x_j is not visible from x_k; $i, j, k \in \{1,2,3\}$, $i \neq j \neq k$.

Shermer called the previous problem *The Hunter's Problem* [107]. In his formulation of the problem, a hunter tries to catch a prey that is trapped within a polygonal region. The prey is considered caught if it comes within sight of the hunter. As do Suzuki and Yamashita, he assumes that the prey can move at an arbitrarily large speed, while the hunter moves with bounded velocity. Urrutia proved:

Theorem 9.1 $O(\ln n)$ *hunters are always sufficient, and occasionally necessary to catch a prey in any polygon with n vertices.*

Let $f(n)$ be the minimum number of hunters needed to catch a prey in an n vertex polygon P. We now show that $f(n) \leq f(\frac{2n}{3}) + 1$. This will prove our result. We first find a diagonal that splits P into two subpolygons P_1 and P_2 of size at most $\frac{2n}{3}$. Station a hunter at an endpoint of this diagonal. This will ensure that no prey can go from P_1 to P_2. Next scan P_1 first, and then P_2. This can be accomplished with at most $f(\frac{2n}{3})$ hunters. Our result now follows. A family of polygons that require $O(\ln n)$ hunters can be obtained from a binary search tree T as follows: first draw T on the plane without crossing edges, and then substitute a sequence of three narrow corridors for each edge of T as in Figure 28.

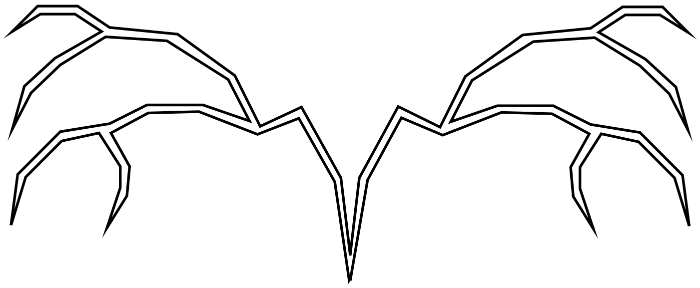

Figure 28: A binary polygon.

9.2 The Two Guard Problem

Icking and Klein [76] introduced the following problem: Let P be a simple polygon, and s and t two points on its boundary. If we place two guards at s, can we move them to t, one in the clockwise direction and the other in the couter-clockwise direction along the boundary of P, while they remain mutually visible all the time? This is called the *Straight Walk Problem.*

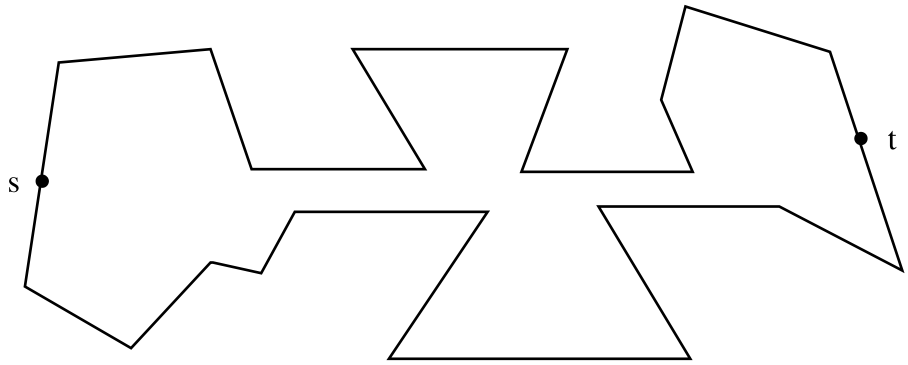

Figure 29: A two-walkable polygon.

Icking and Klein study several variations to the problem. In one of them, the guards are allowed to backtrack. They call this the *General Walk Problem.* They also study the case when one guard moves from s to t while the other moves fron t to s. For the General Walk Problem, Icking and Klein [76] present an $O(n \ln n + k)$ constructive algorithm, where k is the number of times both guards have to backtrack. They show that their algorithm is optimal by showing that computing a general walk has a lower bound of $\Omega(n \ln n)$.

For the straight and the counter walk version of their problem they give $O(n \ln n)$ time algorithms. These results were improved by Heffernan [68] who proved:

Theorem 9.2 *The Straight And Counter Walk Problem can be solved in linear time.*

Heffernan also gave a $\Theta(n)$ time algorithm to *decide* if a polygon with n vertices has a general walk. Notice however that unlike Icking and Klein's optimal $O(n \ln n)$ time optimal constructive algorithm, Heffernan's result does not produce the actual walks that the guards have to walk.

9.3 Lazy guards

Colley, Meijer, and Rappaport [26] studied the following variation of mobile guards, which they call *The Lazy Guard Problem.* Given a polygon P, choose a minimal number of stations (points) in the polygon such that a mobile guard that visits all stations guards P; see Figure 30. They proved:

Theorem 9.3 *An optimal placement of stations for lazy guarding a simple polygon can be found in linear time.*

They also show that the Lazy Guard Problem is NP-complete for polygons with holes.

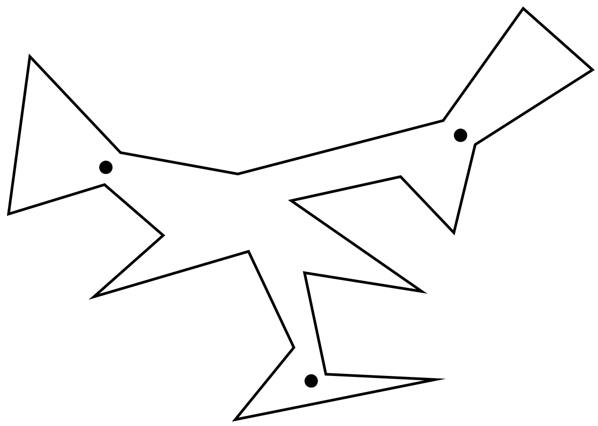

Figure 30: Three guard stations suffice to guard this polygon.

9.4 Treasures in art galleries

Deneen and Joshi [42] studied the following problem: Suppose we have a number of valuable treasures in a polygon P. What is the minimum number of mobile guards required to patrol P in such a way that each treasure is always visible from at least one guard? They show that their problem is NP-hard, and give some heuristic algorithms to solve it.

Carlsson and Jonsson [16] assigned weights to the treasures in the gallery, and asked for the problem of finding the problem of placing a guard in the gallery in such a way that the sum of weights of the treasures visible to the guard is maximized. They show:

Theorem 9.4 *Let P be a polygon with n vertices, and t weighted treasures. Finding the vertex guard that maximizes the weights of the treasures it sees can be done in $O(min\{tn, l \ln n + n^3\})$ time.*

They also study the problem when the treasures are located at vertices of P, on its boundary, or in the interior of P.

10 Miscellaneous

10.1 Polyhedral terrains

Let P_n be a point set on the plane. A triangulation of P_n is a subdivision of the convex hull of P_n into a set of triangles with disjoint interiors such that the vertices of all these triangles are elements of P_n. A *polyhedral terrain* P is defined in Bose, Shermer, Toussaint and Zhu [14] to be a triangulation of a point set P_n on the plane. A set S of vertices (resp. set H of edges) guard a polyhedral terrain P if every face of P has a vertex in S (resp. has an endpoint of an edge of H). The following result is due to Rivera-Campo and Everett [50].

Theorem 10.1 *Any triangulation of the plane can be guarded with at most $\lfloor \frac{n}{3} \rfloor$ edge guards.*

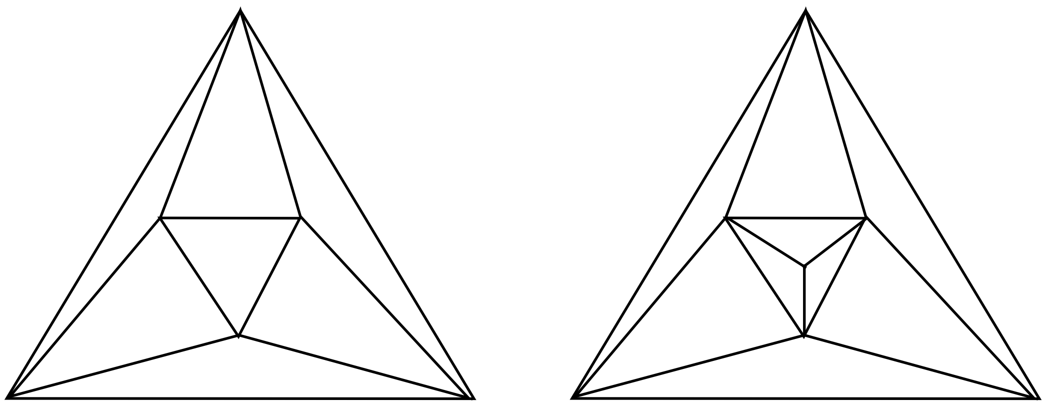

Figure 31: A 6-vertex terrain that needs two edge guards, and a 7-vertex terrain that needs 3 vertex guards.

Their proof uses a nice argument based on the Four Color Theorem. Bose, Shermer, Toussaint and Zhu [14] then proved:

Theorem 10.2 $\lfloor \frac{n}{2} \rfloor$ *vertex guards are always sufficient and occasionally necessary to guard an n-vertex polyhedral terrain.* $\lfloor \frac{4n-4}{13} \rfloor$ *edge guards are sometimes necessary.*

They provide linear time algorithms to place $\lfloor \frac{3n}{5} \rfloor$ vertex guards and $\lfloor \frac{2n}{5} \rfloor$ edge guards respectively. An interesting problem arising from the proof by Everett and Rivera-Campo was to prove Theorem 10.1 without the use of the Four Color Theorem. This was recently achieved by Bose, Kirkpatrick, and Li [13]. They proved Theorem 10.1 using the fact that any bridgeless cubic graph always has a perfect matching. Their proof enabled them to obtain an $O(n^{3/2})$ time algorithm to place at most $\lfloor \frac{n}{3} \rfloor$ edge guards and $\lfloor \frac{n}{2} \rfloor$ vertex guards to guard a polyhedral terrain.

10.2 Protecting convex sets

Another variation to the Art Gallery Theorem was proposed by Czyzowicz, Rivera-Campo, Urrutia and Zaks [36]. In this case, we say that a set S is protected by a guard g if at least one point in the boundary of S is visible from g. The motivation here is that as long as we can see part of an object, it has not been stolen. In [36] they prove the following results:

Theorem 10.3 $\lfloor \frac{2(n-2)}{3} \rfloor$ *guards are always sufficient and occasionally necessary to protect any family of* n *disjoint convex sets,* $n > 2$.

In the same paper, the following results are also proved:

1. $\lceil \frac{n}{2} \rceil$ guards are always sufficient and $\lfloor \frac{n}{2} \rfloor$ are sometimes necessary to protect any family of n isothetic rectangles or any family of homothetic triangles.
2. $\lceil \frac{4n}{7} \rceil$ guards are always sufficient to protect any family of n triangles.
3. There is no constant $c < 1$ such that every family of n convex sets in R^3 can be protected with cn guards.

Once again, most of the proofs are based on Tutte's and Nishizeki's theorems. A family of circles that requires $\lfloor \frac{2(n-2)}{3} \rfloor$ guards is shown in Figure 32.

10.3 Cooperative guards

The concept of *cooperative guards* was proposed by Liaw, Huang and Lee [84]. Let Q be a set of points contained in a polygon P. The visibility graph of Q is the graph with vertex set Q in which two elements of Q are adjacent if they are mutually visible. A set of guards is called *cooperative* if they guard all of P and their visibility graph is connected. The idea behind this concept is that it could be dangerous for a guard to be stationed in an isolated location, unable to communicate with other guards.

The main result in Liaw, Huang and Lee's paper is

Theorem 10.4 *The* Minimum Cooperative Guards Problem *for simple polygons is NP-hard. For spiral and 2-spiral polygons, this problem can be solved in linear time.*

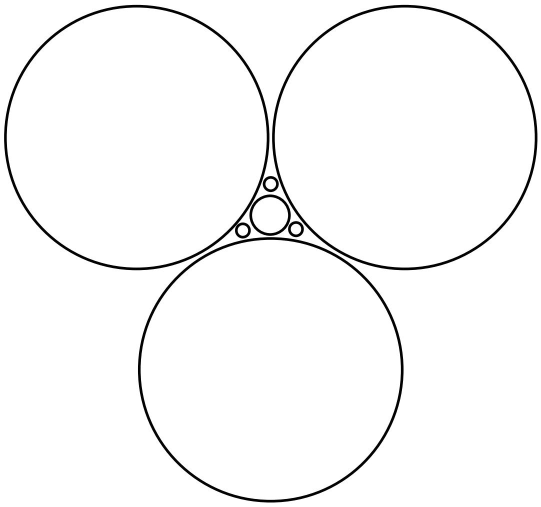

Figure 32: A set of circles that needs $\lfloor \frac{2(n-2)}{3} \rfloor$ guards to protect them.

The proof for NP-hardness follows directly from the proof of Lee and Lin [82] that Theorem 4.1 is NP-complete.

Combinatorial bounds for the minimum number of cooperative guards needed to guard a polygon were given by Hernandez-Peñalver in [71]. He studied what he called *watched* guards, that is, sets of guards in which each guard is "guarded" by at least one other guard. He proved:

Theorem 10.5 $\lfloor \frac{2n}{5} \rfloor$ *watched guards are always sufficient and occasionally necessary to guard a polygon with* n *vertices.*

He then shows that any polygon with n vertices can always be guarded with $\lfloor \frac{2n}{5} \rfloor$ guards. And since cooperative guards are watched guards, he established:

Theorem 10.6 $\lfloor \frac{2n}{5} \rfloor$ *cooperative guards are always sufficient and occasionally necessary to guard a polygon with* n *vertices.*

10.4 Guarding polygons using consecutive edges or vertex sets

The problem of determining if an arc of the boundary $\delta(P)$ of a polygon P guards it has also been studied in the literature. By an *arc* of $\delta(P)$, we understand a contiguous subset of $\delta(P)$ bounded by two arbitrary points, not necessarily vertices of P, on $\delta(P)$.

The first result in this direction was obtained by Avis and Toussaint [6]. They gave an optimal linear time algorithm to determine the visibility region of an edge e of P, that is, the set of points in P visible from at least one point on e. They also studied the problem of finding the set of points visible from *every* point in e. They call the first concept *weak visibility* and the former *strong visibility*. In 1992, Chen [20] developed an optimal parallel algorithm to compute the visibility region of an edge of a polygon. His algorithm works in $O(\ln n)$ time using $O(\frac{n}{\ln n})$ processors in the CREW-PRAM computational model.

Abellanas, García-López and Hurtado [2] study the problem of determining the minimum number of consecutive edges and guards required to cover a polygon P.

Theorem 10.7 *For n even (resp. n odd) $n-3$ vertex guards (resp. $n-4$) are always sufficient and occasionally necessary to guard any polygon with n vertices, $n \geq 4$.*

They also show that $n-5$ consecutive edge guards are always sufficient and occasionally necessary to guard any polygon P with n vertices, $n > 5$. For $n \leq 5$, one edge guard suffices. Their results lead to linear time algorithms.

For the problem of finding the minimum number of consecutive vertex and edge guard sets, they claim an $O(n^2 \log n)$ time algorithm in the same paper. Their algorithm can also be modified to compute the shortest guarding arc of $\delta(P)$ in $O(n^2 \log n)$ time.

These results were improved by Chen, Estivill-Castro and Urrutia [21], who proved:

Theorem 10.8 *The shortest guarding boundary chain and the smallest guarding set of consecutive vertices of a polygon with n vertices can be found in $O(n \ln n)$ time.*

They also gave parallel implementations of their results that run in $O(\log n)$ time using $O(n)$ processors in the CREW-PRAM model.

10.5 K-guarding

The following problem was posed by A. Lubiw at the Fourth Canadian Conference in Computational Geometry: Suppose we want to guard a polygon P using guards stationed in the interior of the edges of P and in such a way that *every* edge of P is assigned *at most one guard*. How many guards are needed to guard P? For security reasons, we may want to cover every point of P by k guards; for what values of k is it possible to k-guard P? Shermer noticed that the comb polygon showed in Figure 33 is not 3-guardable. Belleville, Bose, Czyzowicz, Urrutia and Zaks [9] proved:

Theorem 10.9 *Every polygon can be 2-guarded with at most $n-1$ guards. Every polygon can be 1-guarded with at most $\lfloor \frac{n-1}{2} \rfloor$ guards.*

The polygon in Figure 33 with $2m+1$ vertices requires $2m-2$ guards to be 2-guarded, and $m-1$ guards to 1-guard it. The results in [9] lead to linear time algorithms to find those guards.

10.6 Minimizing and maximizing visibility

The problem of finding the location for a guard inside a polygon in such a way that the area covered by the guard is maximized or minimized has also been studied. Intuitively speaking, if we have only one guard available, the best place to station it is the point with the largest visibility area. On the other hand, if we want to hide a treasure inside a polygon, the best place to place it is the point that minimizes visibility. Ntafos and Tsoukalas [95] proved:

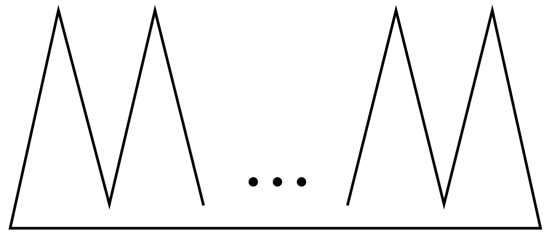

Figure 33: A $2m + 1$ polygon that is 2-guarded by $2m - 2$ guards and 1-guarded by $m - 1$ guards.

Theorem 10.10 *Finding the vertex guard of a polygon P with n vertices that maximizes visibility can be done in $O(n^2)$ time. Finding the point along the boundary of P that maximizes the* area *of visibility can be found in $O(n^3 L)$ where L is the time it takes to numerically solve an equation of degree $O(n)$.*

They also give an approximation algorithm to the boundary problem which in $O(m^2 n^3)$ time produces a solution within $\frac{1}{2m}$ of the optimal one.

The problem of finding the optimal point location of a guard that maximizes the visibility area is open. It is not known if an efficient algorithm to solve it exists.

The problem of finding a point on the boundary of an orthogonal polygon that minimizes visibility was studied by Gewali [58]. He showed:

Theorem 10.11 *The boundary point solution for the Minimum Visibility Problem can be computed in $O(n^3)$ time for orthogonal polygons, and in $O(n^3 \ln n)$ time for orthogonal polygons with holes.*

Gewali also obtains an $O(n^2)$ time algorithm for class 3 orthogonal polygons under staircase visibility. Two points are staircase visible if there is a polygonal joining them consisting of horizontal and vertical line segments such that any parallel to the x- or y-axes intersects it exactly once.

10.7 The Searchlight Problem

Sugihara, Suzuki and Yamashita [113] introduced the *Searchlight Scheduling Problem.* A searchlight is a stationary guarding device based at a fixed position that emits a ray of light in one direction at a time. This direction, though, can be changed, as the searchlight is allowed to rotate continuously around its base. A robber is a point that can move continuously and with unbounded speed. The Searchlight Scheduling Problem is that of determining a way to move a set of searchlights in a polygon P such that an intruder (the "robber") within P can be detected. The intruder is considered detected if at some point in time, it is illuminated by a searchlight. In Figure 34, we show a polygon with two searchlights, as well as the way we move them so that any robber will be detected, no matter how he moves. Sugihara, Suzuki and Yamashita [113] obtained necessary and sufficient conditions under which an intruder can be detected using one and two searchlights. They also claim that in linear time, they can find a set of $\lfloor \frac{n}{3} \rfloor$ searchlights that will always detect an intruder in P. In [114], the same authors study the case when three searchlights are available. Sugihara, Suzuki and Yamashita [113] pose the following open problem:

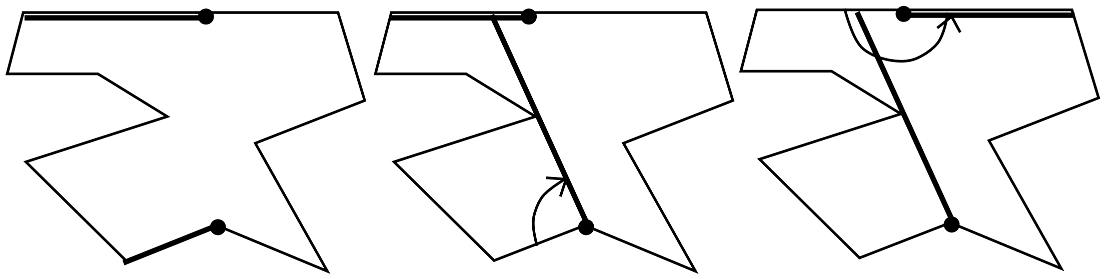

Figure 34: A schedule for two searchlights.

Problem 10.1 *Determine necessary and sufficient conditions for the existence of a search schedule for k searchlights.*

10.8 Coverings and partitionings of polygons

A polygon P is called *star shaped* if there is a point $q \in P$ such that all the points of P are visible from q. The *kernel* of a polygon P is the set of all points that see all of P. It follows by definition that star-shaped polygons are those that can be guarded with one guard. The recognition problem for star-shaped polygons was completely settled by Lee and Preparata [83]. They showed:

Theorem 10.12 *The kernel of a polygon can be found in linear time.*

It is obvious that finding a minimal star shaped covering of a polygon P is equivalent to find the smallest guarding set of P. This leads us to study the following problems:

1. *Convex Covering Of Polygons.* Find the smallest number of convex polygons needed to cover a polygon. Notice that in this case, the polygons need not have disjoint interiors.

2. *Convex Partition Of Polygons.* Partition a polygon into a set of convex polygons with disjoint interiors.

Culberson and Reckhow [29] showed that the Convex Covering Problem is NP-hard. They also showed that the related problems of covering only the edges or vertices of a polygon are also NP-hard. Hecker and Herwing [67] showed that Diagonal Guarding, Triangulation Triangle Guarding, and Mobile Guarding are all NP-hard.

Chazelle and Dobkin [19] proved, surprisingly, that the problem of finding a minimum convex partitioning of a polygon can be done in polynomial time. They proved:

Theorem 10.13 *Finding a minimum convex partitioning of a simple polygon P with n vertices into convex pieces can be done in $O(n + N^3)$ time, where N is the number of reflex vertices of P.*

Lingas [85] has shown that this problem becomes NP-hard for polygons with holes. Keil [78] studied the problem of partitioning a polygon into a set of star-shaped polygons and convex polygons such that the vertex set of each of these polygons is a subset of the vertices of the original polygon. He proved:

Theorem 10.14 *Finding a minimum star-shaped (resp. convex) partitioning of a simple polygon P with n vertices can be done in $O(n^5 N^2 \log n)$ time, (resp. $O(N^2 n \ln n)$) where N is the number of reflex vertices of P.*

In the same paper, Keil also studies partitioning problems in which the length of the internal edges used is minimized. Aggarwal, Ghosh, and Shyamasundar [3] gave another approximation algorithm to partition a polygon into *restricted star-shape* polygons. A restricted star-shaped polygon of a polygon P is a star-shaped subpolygon of P such that its edges are contained in edges or extensions of edges of P. They give an $O(n^4 \ln n)$ time algorithm to find a decomposition of a polygon P into restricted star-shaped subpolygons such that the number of polygons obtained is within $O(\ln n)$ times of the number of subpolygons in an optimal partition.

The problem of partitioning a polygon into a small number of convex pieces has been studied by Shermer and by Belleville. Shermer proved that deciding if a polygon can be covered by two convex sets can be done in linear time [108], and Belleville [8] proved that the same bound is achievable for 3-convex coverings.

Liu and Ntafos [86] have studied the problem of partitioning orthogonal polygons into star shaped polygons. For monotone orthogonal polygons, they were able to solve this problem in linear time. They used this as a basis to obtain an $O(n \log n)$ approximation algorithm to partition a simple orthogonal polygon into star-shaped pieces such that the solution obtained is within six times the size of the optimal partitioning.

Culberson and Reckhow also showed [30] that finding a minimum *rectangle cover* of an orthogonal polygon is NP-complete. Conn and O'Rourke [27] showed that the Rectangle Cover Problem for orthogonal polygons with holes is NP-complete even if we are interested in covering only the reflex vertices. In the same paper, they showed that finding a minimum rectangle cover for the set of convex vertices of orthogonal polygons can be done in $O(n^{2.5})$. It is easy to see that this case reduces to that of finding a maximum matching in a bipartite graph with vertex set equal to the set of convex vertices of an orthogonal polygon, two of which are adjacent iff they are rectangularly visible.

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