A Combinatorial Property of Convex Sets

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Abstract

A known result in combinatorial geometry states that any collection P_n of points on the plane contains two such that any circle containing them contains n/c elements of P_n, c a constant. We prove: Let $\mathcal{F}$ be a family of n disjoint compact convex sets on the plane, S be a strictly convex compact set. Then there are two elements S_i, S_j of $\mathcal{F}$ such that any set S' homothetic to S that contains them contains $\frac{n}{c}$ elements of $\mathcal{F}$, c a constant (S' is homothetic to S if $S'=\lambda S + \mathbf{v}$, where λ is a real number greater than 0 and **v** is a vector of $\mathbb{R}^2$). Our proof method is based on a new type of Voronoi diagram, called the "closest covered set diagram" based on a convex distance function. We also prove that, unlike previous generalizations of the original result on points and circles, our result does not generalize to higher dimensions, we construct a set $\mathcal{F}$ of n disjoint convex sets in $\mathbb{R}^3$ such that for any subset H of $\mathcal{F}$ there is a sphere S_H containing all of the elements of H, and no element of $\mathcal{F}$-H is contained in S_H.

1. Introduction

A known result in combinatorial geometry asserts that any collection P_n of n points on the plane contains two elements u, v $\in P_n$ such that any circle containing them contains at least $\frac{n}{c}$ points of P_n. The first proved value for c was 60 [9], which was successively improved to 30 [2], then to $\frac{84}{5}$ [6] and at this point, the best known value for c is 4.7 (see [3]). Containment problems between families of points and circle, originated from the study of circle orders, i.e. partial orders obtained from containment relations of families of circles on the plane, see [4, 10].

In this paper we prove the following: Let S be a strictly compact convex set (i.e. it is closed and bounded and has no piecewise linear segments on its boundary) and $\mathcal{F}=\{S_1,...,S_n\}$ a family of disjoint compact convex sets. Then there are two elements S_i, $S_j \in F_n$ such that any set S' homothetic to S containing them contains $\frac{n-2}{30}$ elements of $\mathcal{F}$. (S' is homothetic to S if $S'=\lambda S + \mathbf{v}$, where λ is a real greater than 0 and **v** is a vector of $\mathbb{R}^2$). Our proof relies heavily on a new type of *Voronoi diagrams*, which we suggest to call the "closest covered set Voronoi diagram". In general, a Voronoi diagram is a partitioning of the plane according to a distance function defined between the points of the plane and a collection of sites (the elements of $\mathcal{F}$) according to a distance function defined by S.

The original result for points and circles proved in [9] has been generalized to higher dimensions in [2]; in [1] a surprising generalization using collections of points and ellipsoids in euclidean spaces was given. Our new result seems to be of a different nature that the previous ones: it does not generalize to higher dimensions. An example of a family H' of n disjoint convex sets in $\mathbb{R}^3$ with the property that for every pair of elements of H' there is a sphere that contains them and no other element of H' is presented at the end of our paper.

2. Proof of our main result

Our objective in this section is to prove:

Theorem 1: *Let* $\mathcal{F} = \{S_1,...,S_n\}$ *be a family of disjoint compact convex sets on the plane, and* S *any strictly convex compact set. Then there are two elements* $S_i, S_j \in \mathcal{F}$ *such that any convex set* S' *homothetic to* S *containing them contains at least* $\frac{n}{30}$ *elements of* $\mathcal{F}$.

In order to prove our main result, we will need the folowing:

Theorem 2: *Let* H *be any family of five disjoint compact convex sets,* S *a strictly convex compact set. Then there are two elements* $S_i, S_j \in H$ *such that any* S' *homothetic to* S *containing them contains another element of* H.

To avoid getting lost in details, and concentrate on the main ideas behind our result, we will prove Theorem 2 in section 3. We now proceed to prove Theorem 1.

Proof of Theorem 1: Construct a bipartite graph G with $V(G) = X \cup Y$ where X consists of all subsets of pairs of elements of $\mathcal{F}$ and Y contains all of the subsets of $\mathcal{F}$ with exactly five elements. A vertex $T \in X$ is adjacent to a vertex $T' \in Y$ iff $T \subset T'$ and any homothetic copy of S containing the elements of T contains at least another element of T'. By Theorem 2, the degree of every element $T' \in Y$ is at least one and thus the sum of the degrees of all elements in X is at least $|Y| = \binom{n}{5}$. Then there is a vertex in X with degree at least

$$\frac{|Y|}{|X|} = \frac{\binom{n}{5}}{\binom{n}{2}} = \frac{(n-2)(n-3)(n-4)}{60}$$

In other words, there exists a subset T of $\mathcal{F}$ containing two elemets, say S_i and S_j such that any set S' homothetic to S that contains them, contains at least one other element of $\frac{(n-2)(n-3)(n-4)}{60}$ five subsets of $\mathcal{F}$. Allowing for redundancies (each T, together with any other element of $\mathcal{F}$, belongs to $\frac{(n-3)(n-4)}{2}$ different five subsets of $\mathcal{F}$), we obtain that any S' homothetic to S containing S_i and S_j contains at least $\frac{(n-2)}{30}$ elements of $\mathcal{F}$.

3. Proof of Theorem 2

In this section we prove Theorem 2. Some definitions and terminology will be needed before we can start our proof.

3.1 A generalization of Voronoi diagrams

Suppose without loss of generality that S contains the origin $(0,0)$ in its interior. Let $\mathcal{S} = \{\lambda S = \{\lambda x : x \in S\} : \lambda \geq 0\}$. All elements $\lambda S \in \mathcal{S}$ are homothetic to S. Call $(0,0)$ the vertex of $\mathcal{S}$. In turn we call $(0,0)$ the vertex of λS, $\lambda S \in \mathcal{S}$.

Let S' be homothetic to S. Then S' is a translation by a vector $t = (a,b)$ of some $\lambda S \in \mathcal{S}$, that is $S' = t + \lambda S$; $\lambda S \in \mathcal{S}$, $t = (a,b)$. The vortex of S' is now defined to be the image of $(0,0)$ under this translation, ie. vortex$(S') = t$.

Given any point p on the plane and a convex set Q, define the distance $d_S(p,Q)$ to be the smallest λ such that $S' = p + \lambda S$ and S' contains Q. Given two disjoint sets S_1 and S_2, we may now define the vortex bisector $b_S(S_1,S_2)$ to be the set of points p satisfying:

$$d_S(p,S_1) = d_S(p,S_2)$$

It is easy to see that under the restrictions imposed on S, the vortex bisector of S_1 and S_2 is well defined and is always a simple curve that partitions $\mathbb{R}^2 - b_S(S_1,S_2)$ into two disjoint sets, the one consisting of all points closer to S_1 than to S_2 and the other containing those closer to S_2 than to S_1.

Consider now a family $\mathcal{S}=\{S_1,...,S_n\}$ of disjoint compact convex sets. We may associate to each member of $S_i \in \mathcal{S}$ a Voronoi region $Vor_S(S_i)$ consisting of all points p on the plane such that $d_S(p,S_i) \leq d_S(p,S_j)$ for every $S_j \in \mathcal{S}$, $j \neq i$.

The set of regions $Vor_S(S_i)$ thus obtained will be called the "closest covered set Voronoi diagram" of $\mathcal{S}$ with respect to S and will be denoted by $Vor(S,\mathcal{S})$.

It is easy to verify that if S is a disk C (a circle together with its interior), the origin is the centre of D and $\mathcal{S}$ a set of points, then we obtain precisely a Voronoi diagram.

There are, however, some important differences between $Vor(S,\mathcal{S})$ and regular Voronoi diagrams. It might happen, for example, that there are elements of $\mathcal{S}$ such that $Vor_S(S_i) = \emptyset$. For example if $\mathcal{S} = \{S_1 = \{(-1,0)\}, S_2 = \{(x,y)\text{: } x=0, -2 \leq y \leq 2\}, S_3 = \{(1,0)\}\}$ and C is a disk with center at the origin, then $Vor_C(S_2) = \emptyset$. See Figure 1.

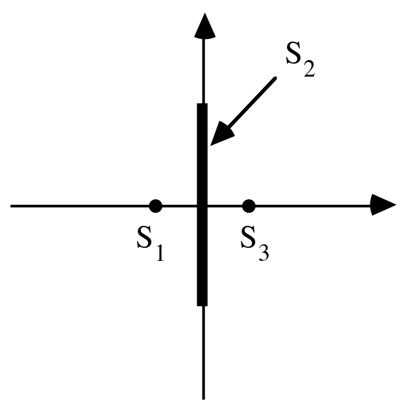

Figure 1.

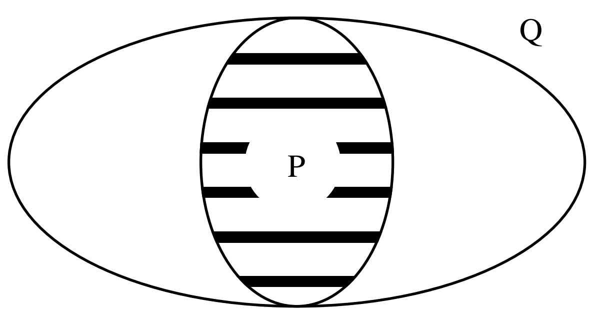

Figure 2

Given two convex sets, P, Q such that $P \subset Q$ we say that P splits Q if Q - P is not connected; see Figure 2.

We proceed now to characterize those sets S_i □ □ for which $Vor_S(S_i) = Ø$. The following observation will be useful:

Observation 1: Let S' and S" be two different strictly convex and homothetic sets. Then their boundaries intersect in at most two points.

Lemma 1: $Vor_S(S_i) = Ø$ *iff there is* S' *homothetic to* S *such that*

i) S_i *splits* S'

ii) *There are two different components* A, B *of* S' - S_i *and two elements* S_j, S_k *of* □ *such that* S_j □ A, S_k □ B. *See Figure 3.*

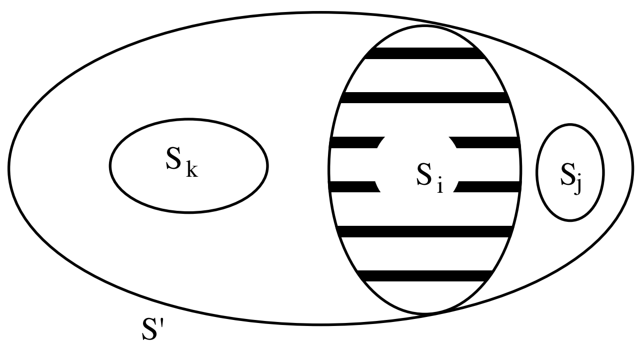

Figure 3

Proof: Suppose that $Vor_S(S_i) = Ø$. Among all S' homothetic to S containing S_i choose the one with smallest area, call it S". It is easy to see that under the conditions assumed on S that S_i splits S". Since $Vor_S(S_i) = Ø$ the vortex v" of S" belongs to $Vor_S(S_j) = Ø$ for some $j \neq i$. This implies that S_j is contained in S", and since S_i splits S" it belongs to one component H of S"-S_i. Assume for the moment that no other element of □ is contained in H (this restriction can be easily deleted and, assuming it true simplifies considerable our proof). Using tools in mathematical analysis, it is easy to show that there is a set S' homothetic to S that satisfies the following conditions:

a) S' contains S_i and S_j.

b) S_i splits S'

c) S_j intersects the boundary of S'.

It is now easy to see using continuity arguments that if all elements S_k with $k \neq i,j$ are not contained in S' then we could easily find a homothetic copy Q of S that contains only S_i. (See figure 4). That is the vortex of Q belongs to $Vor_S(S_i)$, which is a contradiction.

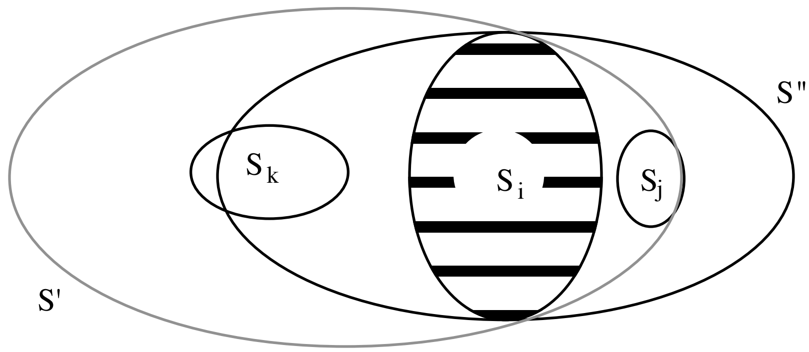

Figure 4.

Conversely, suppose that there is a set S' satisfying the conditions of our lemma. Then it is easy to see using observation 1 that any homothetic copy S'' of S containing S_i contains either S_j or S_k in its *interior*. In either case, the vortex of S'' belongs to $Vor(S_i)$ or $Vor(S_j)$ respectively, and thus $Vor(S_i)=\emptyset$.

We shall also need the following result:

Lemma 2: *If* $Vor_S(S_i) \neq \emptyset$, *then it is also connected.*

The proof of this lemma is interesting on its own. To avoid breaking the flow of our paper we will postpone the proof of this lemma until section 3.3.

Next we prove:

Lemma 3: *Let* $\mathcal{S}$ *be a family with at least four convex sets, and let* $S_i \in \mathcal{S}$ *such that* $Vor_S(S_i) = \emptyset$. *Then there is* $S_j \in \mathcal{S}$ *such that any* S' *homothetic to* S *containing* S_i *and* S_j *contains other elements of* $\mathcal{S}$.

Proof: By Lemma 1, since $Vor_S(S_i) = \emptyset$, there are two elements of $\mathcal{S}$, say S_1 and S_2 different from S_i, and a homothetic copy S' of S such that S_i splits S' and S_1 and S_2 are in two different components, A and B of $S' \setminus S_i$ respectively.

Since $\mathcal{F}$ has at least four elements, there is $S_j \in \mathcal{F}$ different from S_1, S_2 and S_i. We now prove that any S'' homothetic to S containing S_i and S_j contains S_1 or S_2.

Call γ_A the section of the boundary of S' that also belongs to the boundary of A. Suppose then that there is a homothetic copy of S'' that contains S_i and S_j and does not contain S_1. Then S'' intersects γ_A in at least two points (see Figure 4). It now follows that B, and hence S_2 is totally contained in S''. Similarly we can prove that if S_2 is not contained in S'' then S_1 is, and our lemma is proved.

We are now ready to prove Theorem 2:

Proof of Theorem 2: Consider a family of five disjoint compact convex sets $\mathcal{F}=\{S_1,...,S_5\}$ and a strictly convex compact set S. Consider the Voronoi diagram Vor(S,$\mathcal{F}$). By Lemma 3 we can assume that $Vor(S_i)\neq\emptyset$ i=1,...,5. Construct a graph G in which for every region of Vor(S,$\mathcal{F}$) there is a vertex in G. Two vertices of G are adjacent if their corresponding Voronoi regions are adjacent. Then by planarity arguments there are two different values i and j such that $Vor(S_i)$ and $Vor(S_j)$ are not adjacent, otherwise a planar embedding of the complete graph on five vertices would be obtained. It is now immediate that any S' homothetic to S containing both S_i and S_j contains at least another element S_k of $\mathcal{F}$, that is the set S_k containing the vortex of S'.

3.2 Dimension 3

We now present an example of a family $\mathcal{F}$ of n disjoint convex sets in $\mathbb{R}^3$ for which for every subset $\mathcal{F}'$ of $\mathcal{F}$ there is a sphere that contains all of the elements of $\mathcal{F}'$ and does not contain any other element of $\mathcal{F}$. First we construct a set $Q=\{P_1,...P_n\}$ of n polygons on the plane as follows: Consider a circle C on the x-y plane of $\mathbb{R}^3$. For each non-empty subset H of $I_n=\{1....,n\}$ choose an interval C_H of C such that if H and H' are different subsets of I_n then $C_H \cap C_{H'} = \emptyset$. Let C^-_H and C^+_H be the initial and final points of C_H in the counterclockwise direction. For every non empty subset $H=\{i(1),...,i(k)\}$ of I_n take n equidistant points in the interior of C_H and label the first k of them in the counterclockwise direction with the integers i(1),...,i(k) and the remaining n-k points with the integers in I_n-{i(1),...,i(k)} (See figure 5).

Let p_H be any point in C_H to the right of i(k) appearing before any of the points on C_H with a label in I_n-{i(1),...,i(k)}. For every i let P_i be the convex closure of all of the 2^n -1 points in C labelled i.

It is easy to see that any circle that intersects C at C^-_H and □$_H$ and contains the arc of C from C^-_H to □$_H$ contains all of the polygons P_i, i□H and does not contain any polygon P_j such that j □ H. For every i, let P'_i□ be a translate of P_i in the direction of the z-axes by a small □$_i$, i=1,...,n ; □$_i$≠□$_j$, i≠j. Let □ = {P'_1□ ,...,P'_n□}. For every subset H={i(1),...,i(k)} of I_n let □$_□$={$P'_{i(1)}$□ ,...,$P'_{i(k)}$□ }. It is now easy to see that for any non empty subset □$_□$ of □ there is a sphere in R^3 that contains only the elements in □$_□$ and contains no other element of □.

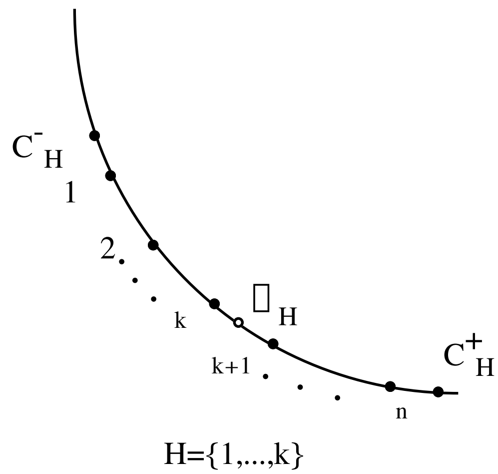

Figure 5

3.3 The proof of Lemma 2

We proceed now to prove that if a set S_i is such that $Vor_S(S_i) \neq \emptyset$ then $Vor_S(S_i)$ is connected. We will actually prove that if the interior $Int(Vor_S(S_i))$ of $Vor_S(S_i)$ is non-empty, then it is connected. This suffices to prove our result.

We tackle first the case when S_i *is a convex polygon*. To this end, let us assume that Q_i □ □ is a convex polygon, We may assume that for this case, the boundary of S(a') contains two vertices of Q_i. Otherwise, by first shrinking S(a') while keeping its vortex fixed, we can assure that the boundary of S(a') touches the boundary of Q_i in at least one vertex, say p. Next, consider the line segment L joining a' to p. For every point x of L let S(x,p) be the homothetic copy of S with vortex x and containing p on its boundary. Clearly all S(x,p) are contained in S(a'). If x is sufficiently close to p S(x,p) does not contain Q_i, and since S(a') contains Q_i there is a point y in L such that S(y,p) contains Q_i and intersects the boundary of Q_i at p and another vertex of Q_i

different from p. Similarly, we can assume that $S(a'')$ intersect the boundary of Q_i in at least two vertices.

Let p and q denote the vertices of Q_i on the boundary of $S(a')$. Assume without loss of generality that the line L through p and q is horizontal. Note that one or both of p and q might be the intersection points of $\partial(S(a')) \cap \partial(S(a''))$.

By assumption, a' belongs to the bisector $b_S(p, q)$ of p and q which is y-monotone since p and q have the same y-coordinate. This follows from a result in [7]. By definition, for each a in $b_S(p, q)$ there is a unique homothetic copy of $S(a)$ of S with vortex a whose boundary contains p and q. Let $S^+(a)$ and $S^-(a)$ be the parts of $S(a)$ above (resp. below) the horizontal line L (See Figure 6).

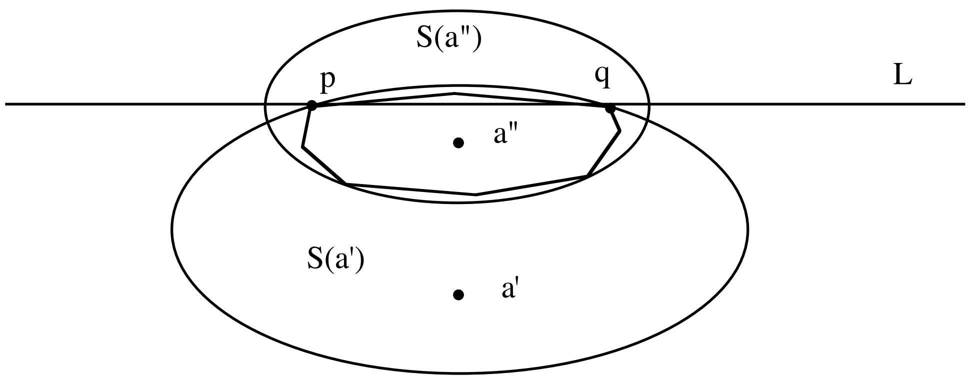

Figure 6

Lemma 4: As a moves along $b_S(p, q)$ in the upward direction, $S^+(a)$ is strictly growing while $S^-(a)$ is strictly shrinking.

Proof: Let a and b be on $b_S(p, q)$. Either $S^+(a)$ contains $S^+(b)$ or viceversa. Suppose the former holds, We have to show that in this case, a lies above b. Let c be the intersection point of the common 'outer' tangents of $S(a)$ and $S(b)$ (See Figure 7).

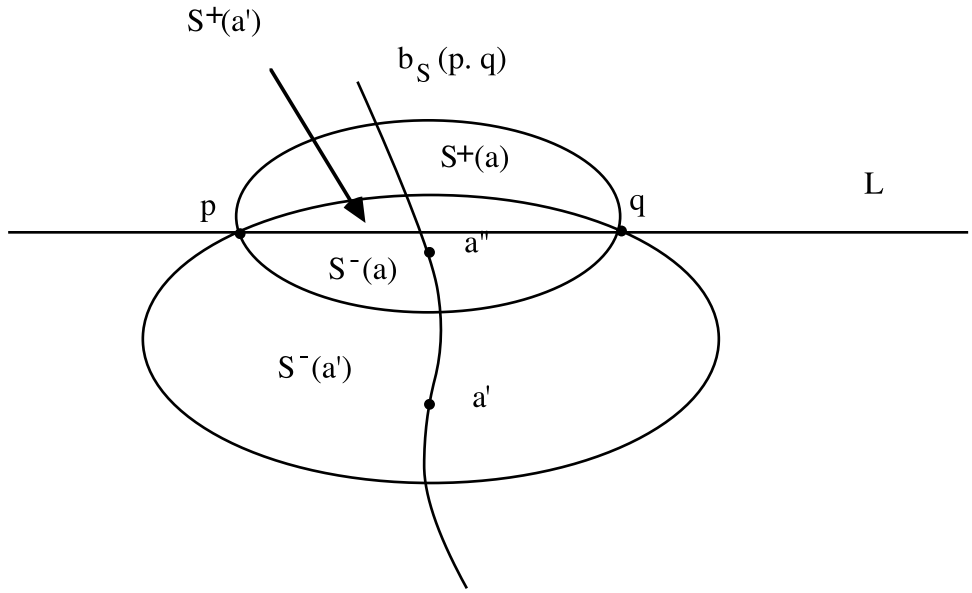

Figure 7

Clearly, c is the center of a homothety between S(a) and S(b). First, assume that c lies above L. From the point of view of c, the tangents intersect first S(a) and then S(b). Consequently, the ray from c through a and b hits a first. This implies that a has a higher coordinate than b. A similar arguments applies if c is below L.

QED.

Now suppose that S(a') and S(a") are such that the boundary of $S^+(a")$ contains $S^+(a')$ (See Figure 7). In order to transform S(a') into S(a") starting at a' we move a point a upwards along $b_S(p, q)$, and study how the set S(a) with vortex at a and whose boundary passes through p and q changes. By lemma 4, $S^+(a)$ grows while $S^-(a)$ shrinks (See figure 8).

Lemma 5: If S(a) contains Q_i, then the only set of □ contained in S(a) is Q_i.

Proof: Suppose that some $S_j \neq Q_i$ is also contained in S(a). Since Q_i and S_j are disjoint, S_j must be contained either in $S^-(a)$ which is contained in S(a') or $S^+(a)$ is contained in S(a"), contradicting our assumptions.

QED.

We can now prove:

Lemma 6: Let Q_i be a convex polygon such that Int($Vor_S(Q_i)$)≠Ø. Then Int($Vor_S(Q_i)$) is connected.

Proof: Let a' and a" are poits in the interior of $Vor_S(Q_i)$ with S(a') and S(a") as defined before. We may assume that each of the boundaries of S(a') and S(a") contains two vertices of Q_i. We have to prove that there is a path from a' to a" entirely contained in Int($Vor_S(Q_i)$). This means that for each point a on this path, a is the vortex of a homothetic copy S(a) of S such that S(a) contains Q_i and no other set $S_j \neq Q_i$. By assumption, S(a') and S(a") satisfy this condition.

There are two possible events that can prevent us from moving a upwards along $b_S(p, q)$. First, it could happen that the shrinking boundary of S(a) hits a third vertex, say r of Q_i (See figure 8). Note that it cannot happen that r is a vertex on the boundary segment of Q_i that runs from the contact points w and v of S(a") with Q_i; this would cause S(a) to intersect S(a") in at least four points, which is impossible, since S(a') and S(a") are homothetic. At this point, we "update" a' and replace it by a.

Now p and q are not the extreme points of Q_i contained on the part of ∂(S(a')) contained in S(a"), these are now r and p. Now we continue moving a along the bisector $b_S(r,p)$ in the same way as before. and update a' as we move a. Eventually, the boundary of $S^+(a)$ will hit the boundary of $S^+(a")$. At this point, S(a)=S(a") and our result follows.

QED.

We are now ready to prove Lemma 2.

Proof of Lemma 2. We will prove that int($Vor(S_i)$) is connected. By lemma 6 we can assume that S_i is not a convex polygon. Let a' and a" points in the interior of $Vor(S_i)$. This implies that there are sets S(a') and S(a") with vortices a' and a" such that S_1 is contained in the interior of S(a') and S(a"). Let Q_i be a convex polygon containing S_i such that Q_i is contained in S(a') ∩ S(a"). Since S_i is a subset of Q_i then $Vor(Q_i)$ is contained in $Vor(S_i)$. By lemma 6, there is a path from a' to a" totally contained in int($Vor(Q_i)$) which in turn is contained in $Vor(S_i)$. Our result now follows.

QED.

4. Conclusions and open problems

We have proved that for any $\mathcal{F}$ of disjoint convex sets, and S convex, there are two elements $S_i, S_j \in \mathcal{F}$ such that any homothetic S' containing S_i and S_j contains $\frac{(n-2)}{30}$ elements of $\mathcal{F}$. We believe that the bound stated in Theorem 1 is far from optimal. At this point we are unable to give a good estimate for an optimal solution to our problem, but we believe that a bound of about $\frac{n}{4}$ or $\frac{n}{5}$ is achievable.

For points and circles, it has been proved that if the elements of P are vertices of a convex polygon, then there are two points of P such that any circle containing them contains $\lceil \frac{(n-3)}{3} \rceil$ points of P, and this is optimal [5]. This result does not extend, however, to the natural generalization to our problem. A family $\mathcal{F}$ of sets is called convexly independent if no element S_i of $\mathcal{F}$ is contained in the convex closure of $\mathcal{F} - S_i$. We have an example consisting of $4n$ elements such that for any pair of elements of $\mathcal{F}$ there is a circle containing them that contains at most $\frac{n}{4}$ elements of $\mathcal{F}$ (see Figure 8).

In this more restricted case we venture the next conjecture:

Conjecture: Let $\mathcal{F}$ be a family of n convexly independent sets, S a convex set. Then there are two elements of $\mathcal{F}$ such that any S' homothetic to S containing them contains at least $\frac{n}{4} \pm c$ elements of $\mathcal{F}$, c a constant.

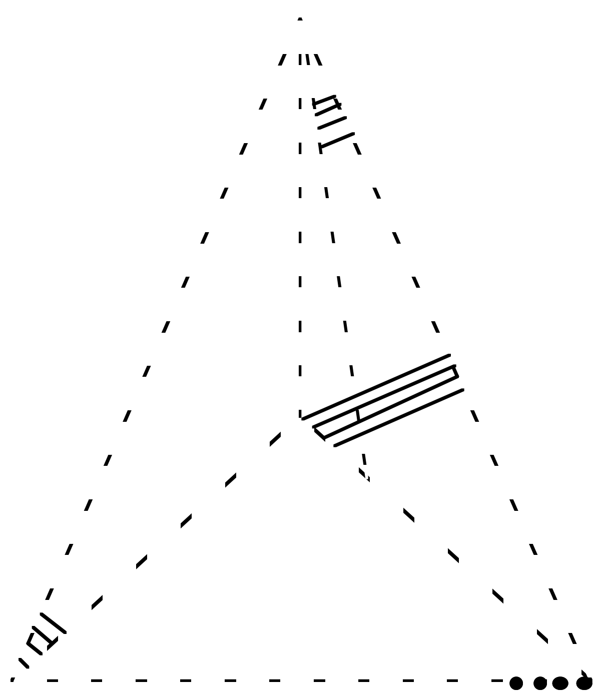

Figure 8

Finally, we mention that the construction of the new type of Voronoi diagrams introduced her, can be carried out in k O(n log n) time, using techniques developed in [8], were k is the time it takes to calculate the vortex bisector of two elements of □.

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