On the number of internal and external visibility edges of polygons

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Abstract

In this paper we prove that for any simple polygon P with n vertices, the sum of the number of strictly internal edges and the number of strictly external visibility edges of P is at least $\lfloor \frac{3n-1}{2} \rfloor - 4$.

The *internal visibility graph* of a simple polygon P is the graph with vertex set equal to the vertex set of P, in which two vertices are adjacent if the line segment connecting them does not intersect the exterior of P. The *external visibility graph* of P is defined in a similar way, except that the line segments that generate its edges are not allowed to intersect the interior of P. A visibility edge is called *strictly internal* (resp. *strictly external*) if it is not an edge of P. In this paper we prove the following conjecture of Bagga [1]:

> For any simple polygon P with n vertices, the number of strictly internal visibility edges plus the number of strictly external visibility edges is at least $\lfloor \frac{3n-1}{2} \rfloor - 4$.

In Figure 1 we present a family of polygons that achieve this bound. They have exactly $n-3$ strictly internal visibility edges, and $\frac{n-3}{2}$ strictly external visibility edges.

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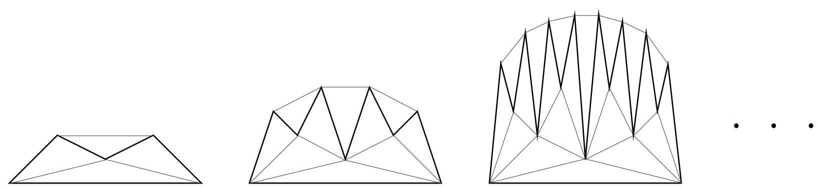

Figure 1: A sequence of polygons for which the number of strictly internal plus strictly external visibility edges is exactly $\lfloor \frac{3n-1}{2} \rfloor - 4$.

Let $int(P)$ and $ext(P)$ denote the number of strictly internal and external visibility edges of P. Some observations will be used to prove that for any polygon P with n vertices, $int(P) + ext(P) \geq \lfloor \frac{3n-1}{2} \rfloor - 4$. A vertex v of P will be called *internal* if it is in the *interior* of the convex hull $Conv(P)$ of P. An *external vertex* is a vertex of the convex hull of P.

The following result is easy to prove:

Lemma 1 *Let P be a simple polygon with n vertices, k of which are internal. Then $ext(P)$ is at least k.*

From this we have:

Lemma 2 *If P has k internal vertices, then $int(P) + ext(P) \geq (n-3) + k$.*

Proof: Observe that any triangulation of P has exactly $n-3$ strictly internal edges. By Lemma 1, $ext(P) \geq k$. ∎

We now prove:

Lemma 3 *If P has k internal vertices, then P can be decomposed into exactly $k+1$ convex polygons $P_1, \ldots, P_{k+1}$. Moreover this decomposition can be achieved in such a way that if n_i is the number of vertices of P_i, $i = 1, \ldots, k+1$, then $n_1 + \ldots + n_{k+1} = n + 3k$.*

Proof: One at a time, and for all the internal vertices v of P, repeat the following operation: starting at v, draw a line segment that bisects the internal angle of P at v and extend it until it hits the boundary of P, or a previously drawn line segment. If this line segment hits a vertex of P, rotate it slightly so that it ends in the middle of an edge; see Figure 2. Observe that the endpoints of these segments, appear as vertices in exactly two of the resulting subpolygons of P, and therefore each of them contributes four units to $n_1 + \ldots + n_{k+1}$. Our result now follows. ∎

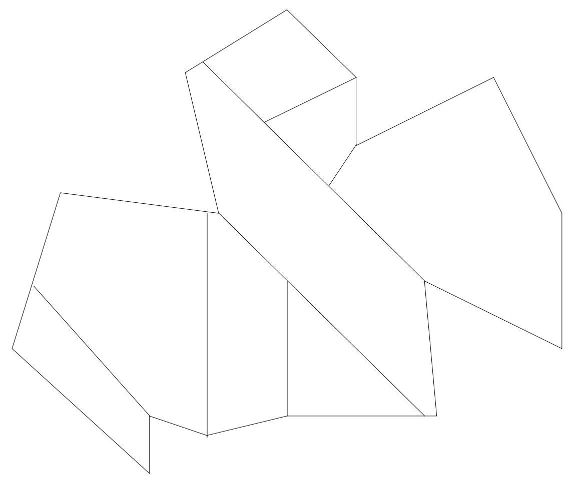

Figure 2: Partitionng a polygon into convex subpolygons.

In our previous lemma, each P_i has two types of vertices; those which are vertices of P, which we call *real* vertices, and vertices which are endpoints of the line segments used to partition P, and which are not vertices of P. Let P_i' be the convex polygon generated by the set of real vertices of P_i, and m_i be the number of vertices of P_i'. Notice that if $m_i \geq 4$, then any *strictly internal visibility edge of* P_i is intersected by at least $m_i - 3$ strictly internal visibility edges of P_i'. Thus we have:

Lemma 4 *If P_i' has m_i vertices, $m_i \geq 4$, then any* strictly internal visibility edge of P_i' *is intersected by at least $m_i - 3$ strictly internal visibility edges of P_i'.*

We now have:

Theorem 1 *For any simple polygon P with n vertices, $int(P) + ext(P) \geq \lceil \frac{3n-1}{2} \rceil - 4$.*

Proof: Suppose that P has k internal vertices. Partition it into $k+1$ convex polygons $P_i, \ldots, P_{k+1}$ as in Lemma 3. If $m_i \geq 4$, select a strictly internal visibility edge e_i of P_i', $i = 1, \ldots, k+1$. Obtain an internal triangulation T of P such that the set of edges e_i as defined before, belong to T. We now show that $int(P) \geq 2n - 2k - 6$. By Lemma 4, for each $m_i \geq 4$, edge e_i is intersected by at least $m_i - 3$ strictly internal edges of P_i'. Since e_i belongs to T, none of these edges belongs to T. Furthermore *these edges are strictly internal visibility edges of P*. It now follows that

$$int(P) \geq (n-3) + \sum_{m_i \geq 4} (m_i - 3) \geq (n-3) + \sum_{i=1,\ldots,k+1} (m_i - 3)$$

But

$$\sum_{i=1,\ldots,k+1} (m_i - 3) = n + k - 3(k+1) = n - 2k - 3$$

(each internal vertex of P appears in two P_i's and each vertex in the convex hull of P in one). Then we have

$$int(P) \geq 2n - 2k - 6$$

By Lemma 1, we know that $ext(P) \geq k$, and thus we have:

$$int(P) + ext(P) \geq 2n - k - 6$$

On the other hand, by Lemma 2 we have that

$$ext(p) + int(P) \geq (n-3) + k$$

Combining these equations we get that $int(P) + ext(P) \geq \lfloor \frac{3n-1}{2} \rfloor - 4$. ∎

References

[1] J. O'Rourke, Combinatorics of visibility and illumination problems. *Technical report* 1996.