On polygons enclosing point sets

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Abstract

Let R and B be disjoint point sets such that $R \cup B$ is in general position. We say that B is enclosed by R if there is a simple polygon P with vertex set R such that all the elements in B belong to the interior of P. In this paper we prove that if the vertices of the convex hull of $R \cup B$ belong to B, and $|R| \leq |Conv(R)| - 1$ then B encloses R. The bound is tight. This improves on results of a previous paper in which it was proved that if $|R| \leq 56|Conv(B)|$ then B encloses R. To obtain our result we prove the next result: Let P be a convex polygon with n vertices $p_1, \ldots, p_n$ and S a set of m points contained in the interior of P, $m \leq n - 1$. Then there is a convex decomposition $\{P_1, \ldots, P_n\}$ of P such that all points from S lie on the boundaries of $P_1, \ldots, P_n$, and each P_i contains a whole edge of P on its boundary.

1   Introduction

Let S be a set of n points in the plane in *general position*. A polygonization of S is a simple polygon such that its vertex set is S. Finding an upper and lower bounds on the number of polygonizations any point set admits is a problem that has been receiving much attention since it was posed in 1979 by Akl [2] and 1980 by Newborn and Moser [11]. In [2] a lower bound of 2.27^n was proved. An upper bound of the form c^n for some constant c (ignoring polynomial terms) was conjectured by Newborn and Moser [11]. This was proved in 1982 by Ajtai, Chvátal, Newborn, and Szemerédi [1], who proved that there are at most 10^{13n} crossing-free graphs on n points in a paper that had strong influence in the latter theory of geometric graphs [3]. This bound has been improved in several papers, and most recently for polygonizations to $c = 86.81^n$ by Sharir and Welzl [13].

Any simple polygon P defines two open regions on the plane, a bounded one called the *interior* of P and an unbounded region called the *exterior* of P. The area of P is the area of the region bounded by P. Problems of finding polygonizations of point sets S that maximize or minimize some parameters of the polygonization have also been studied. S. Fekete [8] considers the problem of finding polygonizations of point sets S that minimize or maximize the enclosed area. He proves that finding such polygonizations is *NP*-complete. It is worth mentioning here that a tool used by Fekete, is Pick's Theorem, a classic result on polygonizations that for polygons with vertices on the integer lattice, establishes an elegant relation between the area of the polygons and the number of lattice points on

the boundary and in the interior of such polygons [12, 4]. The problem of finding polygonizations of point sets that minimize the perimeter is the famous *Euclidean Travelling Salesman* problem, and it is well known that this problem is *NP*-hard [10].
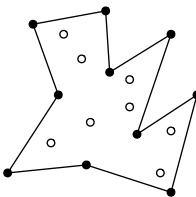

Figure 1: The polygon through the points represented by small solid points encloses all of the points represented by small empty circles.

We say that a polygon P *encloses* a point set S if all the elements of S belong to the interior of P; see Figure 1. Let R and B be disjoint point sets on the plane such that $S = R \cup B$ is in general position. The elements of R (respectively B) will be called the *red* points of S (respectively the *blue* points of S). A polygonization of B will be called a *blue polygonization*. The problem of finding a blue polygonization that encloses as many red points as possible was studied in [5].

Since any red element of S that is enclosed by a blue polygonization must belong to the interior of the convex hull $Conv(B)$ of B, in what follows we will assume that all the elements of R belong to the interior of $Conv(B)$. Under this assumption, it is proved in [5] that there always exists a blue polygonization that encloses at least half of the elements of R. Moreover this bound is asymptotically tight. It is also proved that there always exists a polygon that covers at least half of the area of the convex hull of S.

Let k denote the number of vertices on $Conv(B)$, and i the number of elements of B in the interior of $Conv(B)$, $i + k = n$. In [5] it was also proved that if $|R| \leq \frac{k}{56}$, then there always exists a blue polygonization that encloses R. It is easy to see that there are red point sets contained in $Conv(B)$ with k elements such that the whole of R cannot be enclosed by any blue polygonization; simply let R have an element close enough to the midpoint of each of the k edges of $Conv(B)$, and make sure that B has at least one point in the interior of $Conv(B)$ as shown in Figure 2.
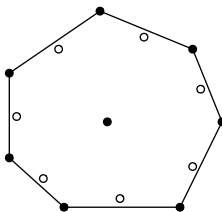

Figure 2: The blue points are represented by small solid circles, the red ones by small empty circles.

Our main goal in this paper is to show that if R has at most $k-1$ points, then there always exists a blue polygonization that encloses R. The main tool used here is a partitioning lemma that we consider to be of interest on its own, asserting the following: Let P be a convex polygon with n vertices, and S a point set contained in P with at most $n-1$ elements. Then there is a set of n convex polygons $P_1, \ldots, P_n$ with disjoint interiors such that the elements of S belong to the boundaries of $P_1, \ldots, P_n$, each P_i contains in its boundary a whole edge of P, and $P_1 \cup \ldots \cup P_n = P$. See Figure 3.

We conclude this paper showing how to construct blue and red point sets with $n = i + k$ and $m = k - 2 + 2i$ elements respectively, such that any blue polygon contains exactly $n - 2$ red points. Observe that when $k = 3$, R has exactly $m = 2n - 5$ elements, and thus any blue polygonization contains exactly $\frac{m+1}{2}$ red points in its interior and $\frac{m-1}{2}$ points in its exterior.
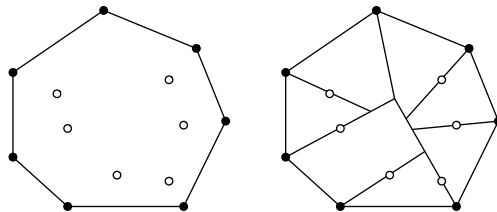
Figure 3: The Decomposition Lemma.

2 The Decomposition Lemma

Let P be a convex polygon with n vertices. We call a set of convex polygons $\{P_1, \ldots, P_n\}$ with disjoint interiors a *convex decomposition* of P if it satisfies the following conditions:

- $P_1 \cup \ldots \cup P_n = P$

- Each P_i has exactly one edge of P on its boundary, called the *lid* of P_i.

$P_1, \ldots, P_n$ will be called the *pockets* of the decomposition. See Figure 3. In the rest of this paper, all point sets or unions of point sets will be assumed to be in general position. If the vertices of a polygon P are labelled $p_1, \ldots, p_n$ in the counter-clockwise order along its boundary, we might refer to P as to the *polygon $p_1 p_2 \ldots p_n p_1$*. In this section we prove:

Theorem 2.1. [**Decomposition lemma**] *Let P be a convex polygon with n vertices $p_1, \ldots, p_n$ and S a set of m points contained in the interior of P, $m \leq n - 1$. Then there is a convex decomposition $\{P_1, \ldots, P_n\}$ of P such that all points from S lie on the boundaries of $P_1, \ldots, P_n$, see Figure 3.*

We present some preliminary results that will be useful to prove Theorem 2.1.

Lemma 2.2. *Let T be a triangle with vertices p_1, p_2 and p_3, that contains in its interior a set S of $2 + x_1 + x_2$ points, where x_1 and x_2 are non-negative integers. Then, there is a point t in the interior of T such that one of the following situations happens:*

(a) *The union of the segments tp_1, tp_2 and tp_3 covers exactly two points from S, and there are x_1 and x_2 points from S in the interior of the triangles tp_1p_2 and tp_2p_3, respectively.*

(b) *Each one of the segments tp_1, tp_2 and tp_3 covers exactly one point from S, and there are $x_1 - 1$ and x_2 points from S in the interior of the triangles tp_1p_2 and tp_2p_3, respectively.*

(c) *Each one of the segments tp_1, tp_2 and tp_3 covers exactly one point from S, and there are x_1 and $x_2 - 1$ points from S in the interior of the triangles tp_1p_2 and tp_2p_3, respectively.*

Proof. Let t_0 be a point on the segment p_1p_3 such that the triangles $t_0p_1p_2$ and $t_0p_2p_3$ have, respectively, $x_1 + 1$ and $x_2 + 1$ points from S in their interior. left).

We consider a point t that moves along the segment t_0p_2, with initial position $t = t_0$, until some point from S is encountered by one or both of the segments tp_1 and tp_3.

If two points are simultaneously found, one by tp_1 and another by tp_3 we are in situation (a).

Suppose then that a point point q_1 from S is intersected by the segment p_1t (the case $q_1 \in p_3t$ is identical). Let t_1 be the intersection point of the lines generated by p_1t and p_2p_3 ; now we move the point t towards t_1 along the line segment p_1t_1. If one point from S is met by either of tp_3 or tp_2 we are again in case (a) and we are done . If two points are simultaneously met by tp_3 and tp_2, we are in situation (c) . □

Proof of Theorem 2.1

Observe that we can assume that $m = n - 1$, for otherwise we can add $(n - 1) - m$ dummy points to the set S, obtain a convex partition, and then remove the dummy points.

We prove our result by induction on n; the base case $n = 3$ follows from Lemma 2.2 with $x_1 = x_2 = 0$. Suppose that the vertices of P are labeled $p_1, \ldots, p_n$ in the counter-clockise direction along its boundary such that the lower vertex of P is precisely p_n, and p_n lies on the origin. Every point q can be described by its polar coordinates $(r(q), \varphi(q))$, where $r(q)$ is the distance from q to the origin and $\varphi(q)$ is the angle from the positive axis $+x$ to the ray through q with apex at the origin.

For every value α in the interval $[0, \pi]$ we define a function g as follows:

$$g(\alpha) = |\{q \in S \mid \varphi(q) < \alpha\}| - |\{p_i \mid \varphi(p_i) < \alpha, 1 \leq i < n\}|.$$

Therefore, in particular,

$$g(\varphi(p_1)) = 0 - 0 = 0;$$
$$g(\varphi(p_{n-1})) = (n - 1) - (n - 2) = 1;$$
$$g(\varphi(p_i)) = |\{q \in S \mid q \text{ is in the interior of the polygon with vertices } p_n, p_1, \ldots, p_i\}| - (i - 1).$$

Let j be the smallest index such that $j > 1$ and $g(\varphi(p_j)) \geq 0$; such an index must exist because $g(\varphi(p_{n-1})) = 1$. Several cases arise.

Case 1: $g(\varphi(p_j)) = 0$. In this case the number of points from S inside the polygon $p_n p_1 \ldots p_j p_n$ is exactly $j - 1$ and we have $g(\varphi(p_i)) < 0$ for all the values of i such that $1 < i < j$, see Figure 4. The
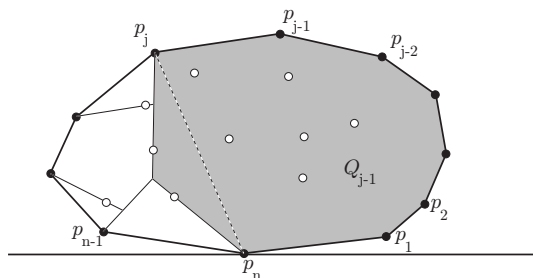

Figure 4: First step in Case 1.

polygon $\widehat{P} = p_n p_j p_{j+1} \ldots p_{n-1} p_n$ has $n - j + 1$ vertices; let $\widehat{S}$ be the set of points of S that are interior to $\widehat{P}$. Since $|\widehat{S}| = (n - 1) - (j - 1) = n - j < n - 1$, we can apply induction to the polygon $\widehat{P}$ and the point set $\widehat{S}$ and obtain a convex partitioning Π of $\widehat{P}$; let $Q \in \Pi$ be the pocket of $\widehat{P}$ whose lid is $p_n p_j$. Let Q_{j-1} be the convex polygon obtained by the union of polygons Q and $p_n p_1 p_2 \ldots p_j p_n$; note that Q_{j-1} contains exactly $j - 1$ points from S, namely the set $S_{j-1} = S \setminus \widehat{S}$.

Let us consider a moving point t that travels counterclockwise on the boundary of Q_{j-1}, starting at p_j. Before t reaches p_n some point $q \in S$ must be meet by the segment $p_{j-1} t$, otherwise we would have $j - 1$ points of S inside $p_n p_1 \ldots p_{j-1} p_n$ and then $g(\varphi(p_{j-1})) > 0$, contradicting the choice of j.

We add the chord of Q_{j-1} through p_{j-1} and q to the decomposition of P that we are constructing and remove from Q_{j-1} the region swept by $p_{j-1} t$ until q was found; in this way we obtain a new convex polygon Q_{j-2} that contains exactly $j - 2$ points from S in its interior, namely the set $S_{j-2} = S_{j-1} \setminus \{q\}$ (Figure 5, left). We repeat the preceding construction by sweeping with a chord of Q_{j-2} having one endpoint anchored at p_{j-2} and so on, until the claimed decomposition of P is completed (Figure 5, right).

Case 2: $g(\varphi(p_j)) > 0$. In this case the number of points from S inside the polygon $p_n p_1 \ldots p_j p_n$ is at least j and we have $g(\varphi(p_i)) < 0$ for all the values of i such that $1 < i < j$.
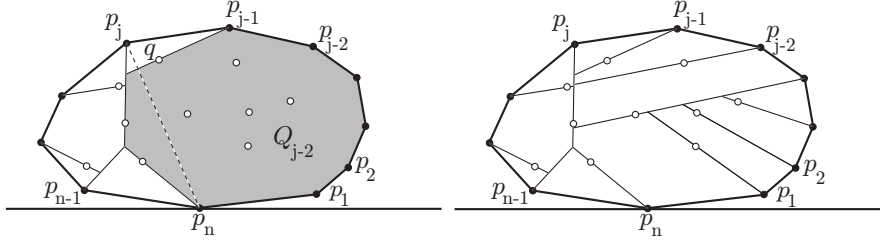
Figure 5: Iterative step and final construction for Case 1.

Let y_1 and y_2 be the number of points of S in the interior of the polygons $p_n p_1 p_2 \ldots p_{j-1} p_n$ and $p_n p_j p_{j+1} \ldots p_{n-1} p_n$, respectively. From the preceding observations we see that $y_1 \leq j - 2$ and that $(n-1) - y_2 \geq j$; i.e., $y_2 \leq n - 1 - j$.

If we define the numbers

$$x_1 = j - 2 - y_1, \ x_2 = n - 1 - j - y_2,$$

we see that the number x of points from S interior to the triangle $p_n p_{j-1} p_j$ is

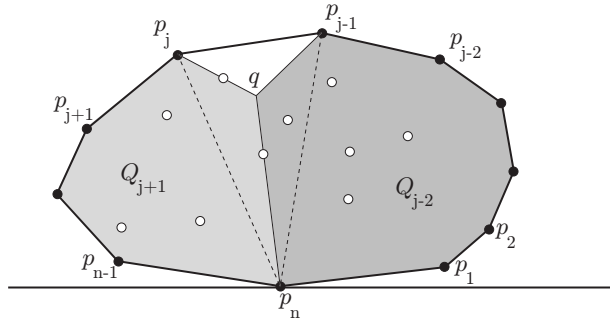$$x = (n-1) - (y_1 + y_2) = (n-1) + (x_1 - j + 2) + (x_2 - n + 1 + j) = x_1 + x_2 + 2 \geq 2.$$

Figure 6: First step for Case 2.

Therefore, we can apply Lemma 2.2 to the triangle $p_n p_{j-1} p_j$ with the numbers x_1 and x_2 associated to the sides $p_n p_{j-1}$ and $p_n p_j$, respectively. Let q be the point such that the segments qp_n, qp_{j-1} and qp_j split the triangle as in Lemma 2.2.

Let Q_{j-2} be the convex polygon obtained as union of the triangle $p_n p_{j-1} q$ and the polygon $p_n p_1 p_2 \ldots p_{j-1} p_n$; let Q_{j+1} be the convex polygon obtained as union of the triangle $p_n q p_j$ and the polygon $p_n p_j p_{j+1} \ldots p_{n-1} p_n$ (Figure 6). Three subcases arise, that we describe separately; in all of them the segments qp_n, qp_{j-1} and qp_j are used for the decomposition of P.

Subcase 2.1: The union of the segments qp_n, qp_{j-1} and qp_j covers two points from S and the triangles $qp_n p_{j-1}$ and $qp_j p_n$ contain x_1 and x_2 points from S, respectively, in their interior.

Let S_{j-2} and S_{j+1} be the set of points interior to Q_{j-2} and Q_{j+1}, respectively. We have $|S_{j-2}| = x_1 + y_1 = j - 2$ and $|S_{j+1}| = x_2 + y_2 = n - 1 - j$ (Figure 7, left).

For the decomposition of Q_{j-2} we consider a segment with an endpoint at p_{j-2} and the other one at a moving point t that travels counterclockwise on the boundary of Q_{j-2}, with starting position $t = p_{j-1}$; a point from S must be found by the sweeping chord before t reaches p_n, and we proceed as in the proof of Case 1 achieving the decomposition of Q_{j-2} (Figure 7, right).
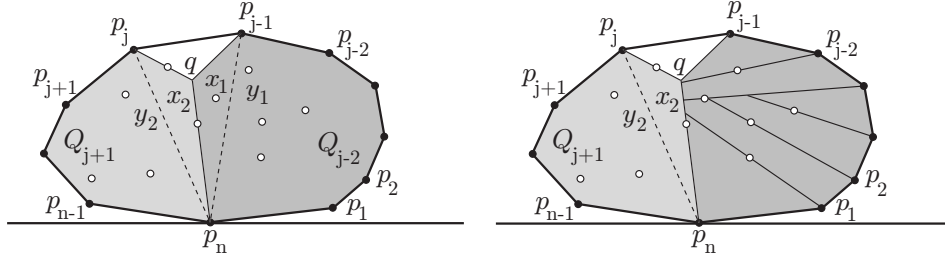
Figure 7: Initial situation in Subcase 2.1 and decomposition of Q_{j-2}.

For the decomposition of Q_{j+1} we consider a segment with an endpoint at p_{j+1} and the other one at a moving point t that travels clockwise on the boundary of Q_{j+1}, with starting position $t = p_j$. If some point of S_{j+1} is found before t reaches p_n, we cut off the swept area and iterate as in the previous situation; if this keeps happening we arrive at the decomposition of Figure 8, left.
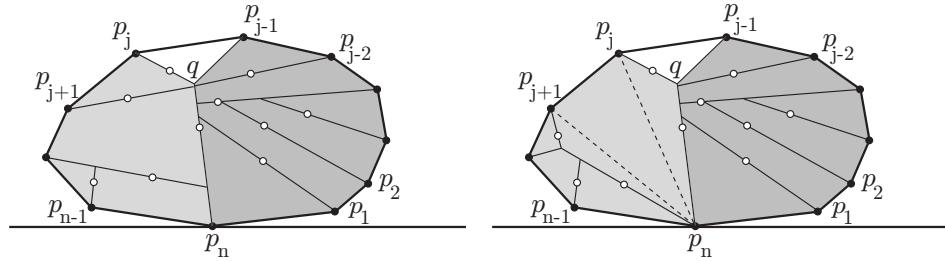

Figure 8: Decomposition of Q_{j+1} in Subcase 2.1.

If t reaches p_n and no point from S_{j+1} has been encountered, all points from S_{j+1} must lie in the interior of the polygon $\widehat{P} = p_n p_{j+1} p_{j+2} \ldots p_{n-1} p_n$; this polygon has $n - j = |S_{j+1}| + 1$ vertices and hence we can apply induction to $\widehat{P}$ and S_{j+1}. In a final step, we obtain the overall decomposition of P by taking as a pocket with lid the edge $p_j p_{j+1}$ the union of the triangles $p_n q p_j$ and $p_j p_{j+1} p_n$ with the pocket corresponding to $p_n p_{j+1}$ in the decomposition of $\widehat{P}$ (Figure 8, right).
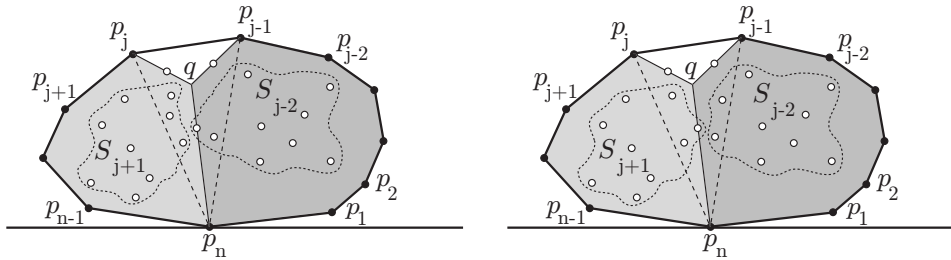

Figure 9: Subcases 2.2 and 2.3.

Subcase 2.2: Each one of the segments qp_n, qp_{j-1} and qp_j covers one point from S and the triangles $qp_n p_{j-1}$ and $qp_j p_n$ contain $x_1 - 1$ and x_2 points from S, respectively, in their interior.

Let S_{j-2} be the set of points of S interior to Q_{j-2} together with the point from S covered by the segment qp_n, and let S_{j+1} be the set of points of S interior to Q_{j+1} (Figure 9, left). We again have $|S_{j-2}| = (x_1 - 1) + y_1 + 1 = j - 2$ and $|S_{j+1}| = x_2 + y_2 = n - 1 - j$ and continue with the sets S_{j-2} and S_{j+1} as in Subcase 2.1 .

Subcase 2.3: Each one of the segments qp_n, qp_{j-1} and qp_j covers one point from S and the triangles qp_np_{j-1} and qp_jp_n contain x_1 and $x_2 - 1$ points from S, respectively, in their interior.

Let S_{j-2} be the set of points of S interior to Q_{j-2} and let S_{j+1} be the set of points of S interior to Q_{j+1} together with the point from S covered by the segment qp_n (Figure 9, right). We again have $|S_{j-2}| = x_1 + y_1 = j - 2$ and $|S_{j+1}| = (x_2 - 1) + y_2 + 1 = n - 1 - j$ and continue with the sets S_{j-2} and S_{j+1} as in Subcase 2.1.

3 Main Result

We recall an observation made in [5].

Observation 3.1. Let P be a convex polygon, p_ip_j an edge of P, and S a set of points in the interior of P. Then there is a simple polygonal starting at p_i and ending at p_j such that its vertex set is $\{p_i, p_j\} \cup S$, see Figure 10.
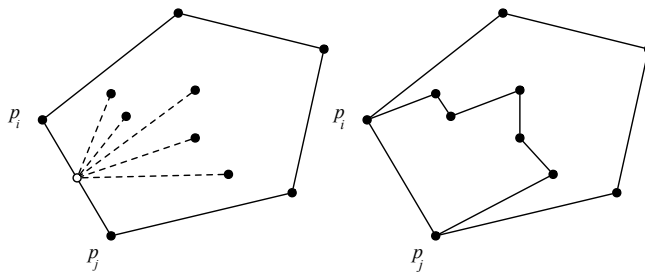

Figure 10: The polygonal within a pocket.

We can now prove:

Theorem 3.2. *Let R and B be two point sets such that $S = R \cup B$ is in general position and R is contained in the interior of $Conv(S)$. Then, if the number of vertices of $Conv(S)$ is k and $|R| < k$, there is a blue polygon enclosing all points of R. This result is tight.*

Proof. By Theorem 2.1, we can obtain a convex decomposition $\{P_1, \ldots, P_k\}$ of $Conv(S)$ such that all the points in S belong to the boundaries of $P_1, \ldots, P_k$. By the previous observation, in each P_i we can find a polygonal that starts and ends at the vertices of P_i in $Conv(S)$ and contains all the elements of B in the interior of P_i. Concatenate the polygonal chains thus obtained. It is clear that this way we obtain a blue polygon that encloses all the elements of R, see Figure 11. $\square$
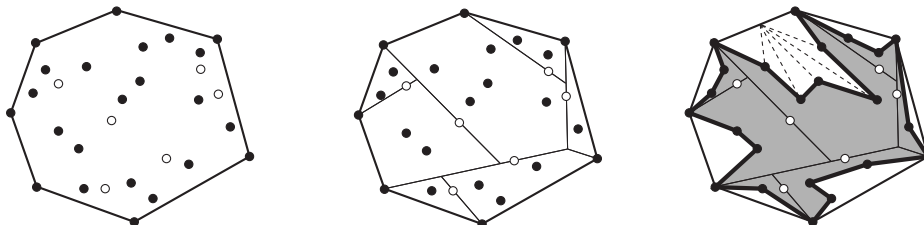

Figure 11: Construction of the enclosing polygonization.

4   Concluding Remarks

The full paper includes some additional results that are omitted from this version due to space limitations. In particular, it is proved that there are sets of points B and R with n and m blue and red points respectively, $R \subset Conv(B)$, such that any blue polygonization encloses $\frac{m+1}{2}$ elements of R. It is also shown that using Pick's theorem one can see that this bound is essentially tight. We refer the interested reader to the complete article.

References

[1] M. Ajtai, V. Chvátal, M. M. Newborn, and E. Szemerédi, *Crossing-free subgraphs*, Annals Discrete Math. 12 (1982), 9-12.

[2] S.G. Akl, *A lower bound on the maximum number of crossing-free Hamiltonian cycles in a rectilinear drawing of Kn*, Ars Combinatoria, volume 7, 1979, pages 7-18.

[3] P. Brass, W. Moser and J. Pach, *Research problems in discrete geometry*, Springer-Verlag, New York, 2005.

[4] H.S.M. Coxeter, *Introduction to geometry*, Wiley, New York, 1969.

[5] J. Czyzowicz, F. Hurtado, J. Urrutia, and N. Zaguia, *On polygons enclosing point sets*, Geombinatorics, Vol. XI-1, pp. 21–28, 2001.

[6] O. Devillers, F. Hurtado, G. Károlyi and C. Seara, *Chromatic variants of the Erdös-Szekeres Theorem*, Computational Geometry: Theory and Applications, 26(3), 2003, 193-208.

[7] A. Dumitrescu and J. Pach, *Partitioning colored point sets into monochromatic parts*, International Journal of Computational Geometry and Applications, 12(5), 2002, 401-412.

[8] S. Fekete, *On simple polygonalizations with optimal area*, Discrete and Computational Geometry, Volume 23 - Number 1, pp. 73–110, 2000.

[9] E. Friedman, *30 Two-Colored Points with No Empty Monochromatic Convex Fourgons*, Geombinatorics, Volume XIV, 2 (2004) 53-54.

[10] M.R. Garey and David S. Johnson (1979). Computers and Intractability: A Guide to the Theory of NP-Completeness. W.H. Freeman, pp.211–212.

[11] M. Newborn and W. O. J. Moser, *Optimal crossing-free Hamiltonian circuit drawings of the K^n*, J. Combinat. Theory, Ser. B 29 (1980), 13-26.

[12] G. Pick, *Geometrisches zur Zahlenlehre*, Zeitschrift des Vereins, "Lotos", Prague, 1899, 311-319.

[13] M. Sharir, nd E. Welzl, *On the Number of Crossing-Free Matchings, Cycles, and Partitions*, SIAM J. Comput. in press.