On balanced 4-holes in bichromatic point sets

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1 Introduction

Let $S = R \cup B$ be a set of n points in general position in the plane. The elements of R and B will be called, respectively, the *red* and *blue* elements of S. A k-hole of S is a simple polygon with k vertices, all in S, and containing no element of S in its interior. A 4-hole of S is balanced if it has two blue and two red vertices. In this paper, we characterize the set of bicolored points $S = R \cup B$ that have balanced *convex* 4-holes. We also show that if the 4-holes of S are allowed to be non-convex, and $|R| = |B|$, then S always has a quadratic number of balanced 4-holes.

The study of k-holes in colored point sets was introduced by Devillers et al. [2]. They obtained a bichromatic point set $S = R \cup B$ with 18 points that contains no convex monochromatic 4-hole. Recently, Hummer and Seara obtained a bichromatic point set with 36 points that does not contain monochromatic 4-holes [3]. This result was improved by Koshelev [4] to 46. Devillers et al. [2] also proved that every 2-colored Horton set with at least 64 elements contains an empty monochromatic 4-hole. In the same paper the following conjecture is posed: Every sufficiently large bichromatic point set contains a monochromatic convex 4-hole. This conjecture remains open. On the other hand, Aichholzer et al [1] proved that any sufficiently large bichromatic point set always contains a *not necessarily convex* monochromatic 4-hole.

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2 Balanced Convex 4-holes

For any two points p and q on the plane, $\overline{pq}$ and $\ell(p,q)$ will denote, respectively, the line segment joining p and q, and the line determined by them. If p and q are both blue points (resp. red points), $\overline{pq}$ and $\ell(p,q)$ will be called a *blue edge*, and a *blue line* of S (resp. *red edge*, and *red line* of S). Given a set X, $CH(X)$ will denote the convex hull of X. In all of our figures, blue points are represented with non-solid small circles, whereas red points are represented with solid small circles. Blue edges will be drawn with dotted line segments, and red edges with solid line segments.

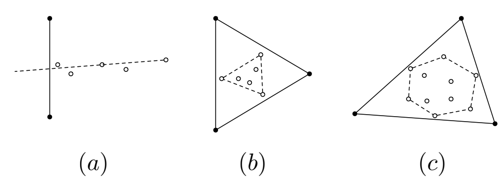

Figure 1: Point sets with no 4-balanced convex holes.

Clearly not all bicolored point sets have a balanced convex 4-hole, see Figure 1(a), (b), and (c). The number of blue points within the blue triangle and hexagon in Figure 1(b), and (c) can be arbitrarily large. We now proceed to characterize bicolored point sets which contain balanced convex 4-holes. We assume that $|R|, |B| \geq 2$.

Lemma 1 *If S contains red and a blue edges that intersect, then S contains a balanced convex 4-hole.*

Proof. Suppose that S has red and blue edges that intersect. Choose a red edge $\overline{ab}$ and a blue edge $\overline{cd}$ such that the convex quadrilateral Q with vertex set $\{a, b, c, d\}$ is of minimum area.

If Q is not a balanced 4-hole of S, then there is a point in S contained in the interior of Q. Suppose w.l.o.g. that Q contains a red point $e \in S$, and that $\overline{ae}$ intersects $\overline{cd}$. Then $\{a, e, c, d\}$ is the set of vertices of a balanced convex quadrilateral with area smaller than that of Q, a contradiction, see Figure 2. □

Two cases arise: R and B are linearly separable, or their are not. We analyze first the case when R and B are not linearly separable.

This is an extended abstract of a presentation given at EuroCG 2012. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear in a conference with formal proceedings and/or in a journal.

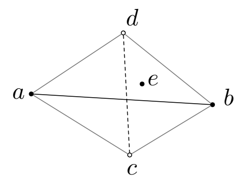

Figure 2: Crossing segments. Segment $\overline{ab}$ can be replaced by segment $\overline{ae}$.

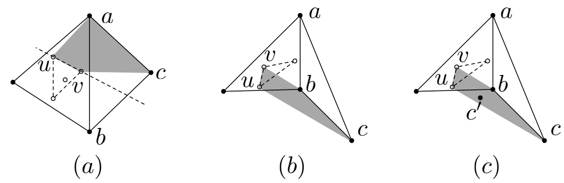

Figure 3: Two possible quadrilaterals including the blue points: convex and non-convex.

2.1 R and B are not linearly separable

Suppose that R and B are not linearly separable. We prove now the next result:

Theorem 2 *Let $S = R \cup B$, such that R and B are not linearly separable. Then S has a balanced 4-hole if and only if either of the following three conditions holds:*

1. *$CH(B) \subset CH(R)$, $|R| \geq 4$, $|B| \geq 2$,*
2. *$CH(R) \subset CH(B)$, $|B| \geq 4$, $|R| \geq 2$,*
3. *$CH(B)$ and $CH(R)$ overlap.*

Proof. Suppose first that $CH(B) \subset CH(R)$, and R and B have at least four and two elements, respectively. Consider a triangulation T of R. By Lemma 1, all blue points have to be in one triangle t of T, for otherwise there would be a red and a blue edge of S that cross each other. If B has exactly two elements, then it is easy to see that two vertices of t, together with the elements of B, form a balanced 4-hole. Suppose then that B has at least three elements.

Since R has at least four points, T has at least two triangles, one of which, call it t', shares an edge $\overline{ab}$ with t. Let $Q = t \cup t'$, and suppose first that Q is convex, see Figure 3(a). If the line determined by two consecutive elements of $CH(B)$, say u and v, intersects $\overline{ab}$, then two vertices of t' together with u and v form a balanced 4-hole, see Figure 3(a).

Suppose then that Q is not convex, and let c be the third vertex of t', see Figure 3(b). Since B has at least three vertices, there are at least two blue vertices on one of the half planes determined by the line $\ell(b, c)$. It is easy to see that we can always choose two of them, call them u and v, such that the quadrilateral Q' with vertices u, v, b, c is convex, and contains no blue point in its interior, see Figure 3(b). It might happen that Q' contains a red point in its interior. In such a case, we can always choose a red point c' in the interior of Q' such that the quadrilateral with vertices u, v, b, c' is a balanced 4-hole, see Figure 3(c). Our result follows. It is worth pointing out that if $CH(B) \subset CH(R)$ but R has only three elements, our result is not true. Counterexamples of this are the point sets in Figure 1(b) and (c).

Finally observe that when the convex hulls of B and R overlap there is a red and a blue edge that intersect. By Lemma 1 we can conclude that S contains a balance 4-hole. $\square$

2.2 R and B are linearly separable.

Suppose that R and B are in convex position, that they are linearly separable, and that there is a line ℓ that separates R and B such that the elements in R are to the left of ℓ, and the elements in B to its right. We will assume w.l.o.g. that ℓ is vertical. Assume also w.l.o.g. that there is a horizontal line ℓ' that is a supporting line of $CH(B)$ and $CH(R)$ that intersects them at a single point, and that R and B are contained in the closed half-plane determined by ℓ' and above it. Let r_1 and b_1 be the lowest elements of R and B respectively. Label the elements of R as $r_1, \ldots, r_m$ in the anti-clockwise direction around $CH(R)$, starting at r_1. In a similar way, label the elements of B as $b_1, \ldots, b_n$ in the clockwise direction around $CH(B)$ staring at b_1.

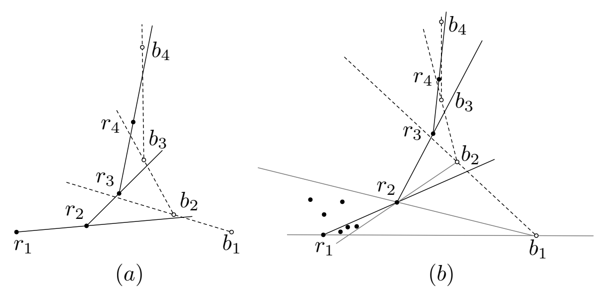

Figure 4: A funnel, and a funnel with a red tail.

We say that $S = R \cup B$ is a *funnel* if the following conditions hold: $\ell(r_1, r_2)$ intersects the segment $\overline{b_1 b_2}$, or $\ell(b_1, b_2)$ intersects the segment $\overline{r_1 r_2}$. In the first case, the following conditions must hold:

a) For each i such that the line $\ell(b_i, b_{i+1})$ and the segment $\overline{r_{i+1} r_{i+2}}$ exist, $\ell(b_i, b_{i+1})$ intersects $\overline{r_{i+1} r_{i+2}}$.

b) For each i such that $\ell(r_i, r_{i+1})$ and $\overline{b_i b_{i+1}}$ exist, $\ell(r_i, r_{i+1})$ intersects $\overline{b_i b_{i+1}}$.

If $\ell(b_1, b_2)$ intersects $\overline{r_1 r_2}$, then $\ell(r_i, r_{i+1})$ intersects $\overline{b_{i+1} b_{i+2}}$, and $\ell(b_i, b_{i+1})$ intersects $\overline{r_i r_{i+1}}$.

The point set in Figure 4(a) is a funnel in which each of R and B contains four elements. It is easy to see that if $R \cup B$ is a funnel, then $||R| - |B|| \leq 1$, and that if we choose a red edge and a blue edge of S, the convex hull of their vertices is a triangle, or it is a convex quadrilateral that contains at least a point of S in its interior.

Suppose that S is a funnel such that $\ell(r_1, r_2)$ intersects $\overline{b_1 b_2}$. Let $\mathcal{T}$ be the unbounded region bounded by $\ell(b_1, r_1)$, $\ell(b_1, r_2)$, and $\ell(b_2, r_2)$. A set of red points R' is called a *red tail* of S if all the elements of R' belong to the interior of $\mathcal{T}$, see Figure 4(b). A blue tail is defined in a similar way when $\ell(b_1, b_2)$ intersects $\overline{r_1 r_2}$.

A *double funnel* is defined as follows: Let R and B be separable point sets in convex position, and let ℓ and ℓ' be as above. Suppose also that there is exactly one edge in the convex hull of R or B such that the line containing this edge separates R and B. Without loss of generality, assume that this edge is red, and that the blue points lie to the right of this line, see Figure 5.

Label the elements of R as $r_1, \ldots, r_n$ in the anti-clockwise direction around $CH(R)$, starting at the lowest element of R. Label the elements of B as $b_1, \ldots, b_m$ in the clockwise direction around $CH(B)$, starting again at the lowest element of B. Suppose that the edge of $CH(R)$ such that the line containing it that separates R and B joins r_i and r_{i+1} for some i.

Let b_i be the closest element of B to $\overline{r_i r_{i+1}}$. Consider the following sets of points: $R_1 = \{r_1, \ldots, r_{i+1}\}$, $R_2 = \{r_i, \ldots, r_n\}$, $B_1 = \{b_1, \ldots, b_i\}$, $B_2 = \{b_i, \ldots, b_m\}$.

We say that $R \cup B$ is a *double funnel* if the sets $B_1 \cup R_1$ and $B_2 \cup R_2$ are funnels, see Figure 5. Note that one of them, is an upside down funnel!

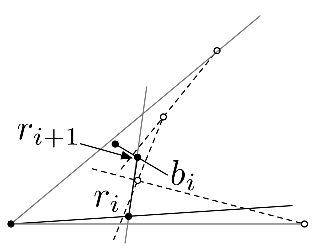

Figure 5: A double funnel.

In a similar way as we defined a tail for a funnel, we can define a tail of a double funnel. A tail of $R \cup B$ is a tail of $B_1 \cup R_1$, or a tail of $B_2 \cup R_2$. The following result is given without proof:

Theorem 3 *If R and B are linearly separable, then $S = R \cup B$ contains no convex balanced 4-holes if and only if S is:*

- *a funnel with or without tail*
- *a double funnel, with one or two tails such that:*
  1. *The two tails have the same color.*
  2. *The double funnel can be splitted into two funnels f_1 and f_2 such that one of them, say f_2 has at most two points of each color.*

3 Non-convex balanced 4-holes

It is not hard to see that if R and B have at least two elements each, then S always contains 4-balanced holes which are not necessarily convex. In fact, it is easy to see that a double funnel contains $O(n^2)$ balanced non-convex 4-holes. In this section we will prove that if $|R| = |B| = n$, then $S = R \cup B$ always contains a quadratic number of balanced, not necessarily convex 4-holes. In the rest of this section, we will assume that $|R| = |B| = n$.

We give our proof only for the case when R and B are linearly separable. Our proof can be modified to prove that for any two points sets R and B, $R \cup B$ always has a quadratic number of balanced holes. We omit these not so trivial changes.

Suppose that there is a horizontal line ℓ that separates R from B, and that the elements in R are above ℓ, and the elements of B below it. We further assume that the y-coordinates of all of the elements of S are different.

Given two points p and q, $p \to q$ will denote the ray staring at p, and passing through q. We now color edges joining blue and red points as follows:

An edge $\overline{rb}$ joining a red point r to a blue point b is colored *green* if it is an edge, or a diagonal of a balanced 4-hole of S. An edge $\overline{rb}$, not colored green, is colored *red* if when we rotate the ray $r \to b$ around r in the clockwise direction, it hits a red point before it hits a blue point. It is not hard to see that in this case, if we rotate $r \to b$ in the anti-clockwise direction, it also hits a red point before it hits a blue point, for otherwise $\overline{rb}$ would be green. Moreover if we rotate $b \to r$ in the clockwise, or anti-clockwise direction, it will also hit red points before it hits blue points. See Figure 6.

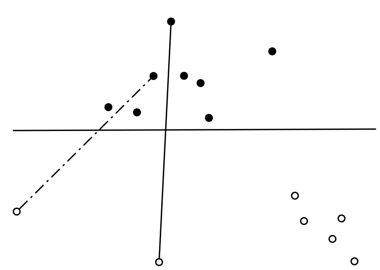

Figure 6: A red and a green edge.

In a similar way we color $\overline{rb}$ *blue* if when we rotate $r \to b$ around b it hits a blue point before it hits a red point. We will prove that the number of green edges is quadratic. Theorem 9 follows easily from this.

Let r be a red point. Sort and relabel the blue points from left to right in the anti-clockwise direction around r as $\{b_1, \ldots, b_n\}$. The following lemmas, are given without proof:

Lemma 4 *There are no consecutive edges $\overline{rb_i}$ and $\overline{rb_{i+1}}$ such that one is blue, and the other is red.*

In other words between a red and a blue edge there is a green edge.

Let i such that $\overline{rb_i}$ and $\overline{rb_{i+1}}$ are both red, and let Δ be the triangle bounded by ℓ, $\overline{rb_i}$, and $\overline{rb_{i+1}}$.

Lemma 5 *Δ contains at least three red points of S, see Figure 7.*

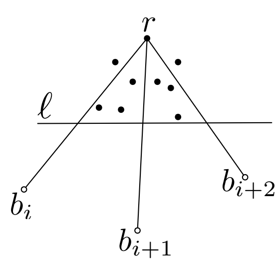

Figure 7

Observe that this Lemma implies that the set $\{\overline{rb_i}; i = 1, \ldots n\}$ contains at most $\frac{n}{3}$ consecutive red edges.

Suppose next that we have a block $\mathcal{B}$ of k consecutive red edges $\overline{rb_i}$, $\ldots$, $\overline{rb_{i+k-1}}$ incident to r, such that $\overline{rb_{i-1}}$ and $\overline{rb_{i+k}}$ are green. Then we can associate to $\mathcal{B}$ at least $3k-1$ red points that lie in the triangle bounded by ℓ, $\overline{rb_i}$, and $\overline{rb_{i+k-1}}$, plus the red points hit by $\overline{rb_i}$ and $\overline{rb_{i+k-1}}$ when we rotate them in the clockwise, and anti-clockwise direction respectively.

It is easy to see that the sets of red points associated to different blocks of consecutive red edges incident to r are disjoint.

Suppose now that the red edges incident to r are grouped into s blocks of consecutive red edges with cardinalities $t_1, \ldots, t_s$.

Label the red points $r_1, \ldots, r_n$ from bottom to top according to their y-coordinate. The next result follows:

Lemma 6 *The number of red edges incident to r_i is at most $(3t_1 - 1) + \cdots + (3t_s - 1) - 2$.*

The worst case is when each $t_i = 1$, in which case the previous Lemma yields $2s - 2$. From here we get:

Lemma 7 *The number of red edges incident to r_i, is at most $\lfloor \frac{i-1}{2} \rfloor + 1$.*

Thus we have:

Theorem 8 *There are at most $2(1 + \cdots \lfloor \frac{n-1}{2} \rfloor) \sim \frac{n^2}{4}$ red edges.*

A simmetric argument shows that the number of blue edges is $\sim \frac{n^2}{4}$.

Since there are exactly n^2 edges joining red and blue points, it follows that at least approximately half of them are green, which proves:

Theorem 9 *The number of balanced bichromatic 4-holes of S is at least $\frac{t}{6}$, were $t \sim \frac{n^2}{2}$.*

Our argument can be modified to prove:

Theorem 10 *Let $S = R \cup B$ be a set with $2n$ points, n red, and n blue. Then S always has at least a quadratic number of balanced bichromatic 4-holes.*

We observe that our bound is asymptotically tight, as in examples as that shown in Figure 8, any balanced 4-hole has to be convex, and there are only a quadratic number of these 4-holes.

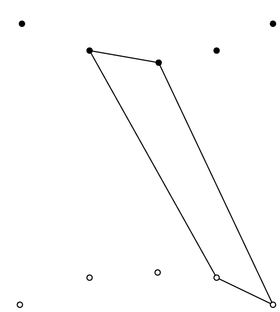

Figure 8: Any balanced 4-hole uses two consecutive red, and two consecutive blue points.

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