Flat 2-foldings of Convex Polygons

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Abstract

A folding of a simple polygon into a convex polyhedron is accomplished by glueing portions of the perimeter of the polygon together to form a polyhedron. A polygon Q is a *flat n-folding* of a polygon P if P can be folded to exactly cover the surface of Q n times, with no part of the surface of P left over. In this paper we focus on a specific type of flat 2-foldings, flat 2-foldings that *wrap* Q; that is, foldings of P that cover both sides of Q exactly once. We determine, for any n, all the possible flat 2-foldings of a regular n-gon. We finish our paper studying the set of polygons that are flat 2-foldable to regular polygons.

1 Introduction

A folding of a simple polygon into a convex polyhedron is accomplished by glueing portions of the perimeter of the polygon together to form the polyhedron (Figure 1). The paper [1] proves the existence of nondenumerably infinite foldings of simple polygons to convex polyhedra. In [2, 5, 3, 4], all possible foldings of an equilateral triangle, square, regular pentagon, and regular n-gons, respectively, are determined, $n \geq 6$. This paper deals with related constructions, flat n-foldings of convex polygons to other convex polygons. Let P and Q be two polygons. We say that Q is a *flat* 2*-folding* of P if P can be folded to wrap Q such that each of its points on both sides of Q is covered exactly once. For example, in Figure 1 we show a flat 2-folding of a pentagon that wraps a triangle.

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In this paper, we focus on flat 2-foldings. In Section 2, we determine all the convex polygons that can result from flat 2-foldings of regular polygons. In Section 3, we determine all the convex polygons that can be flat 2-folded to regular polygons. We conclude the paper with some remarks and open problems.

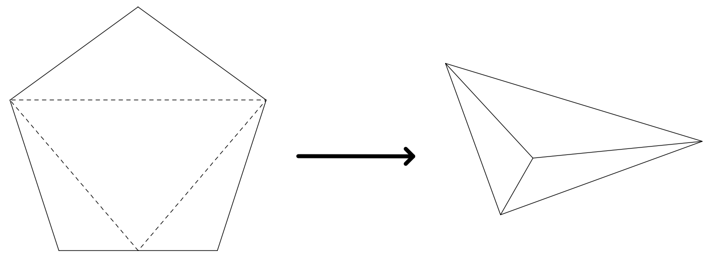

Figure 1: Folding a pentagon to wrap a triangle.

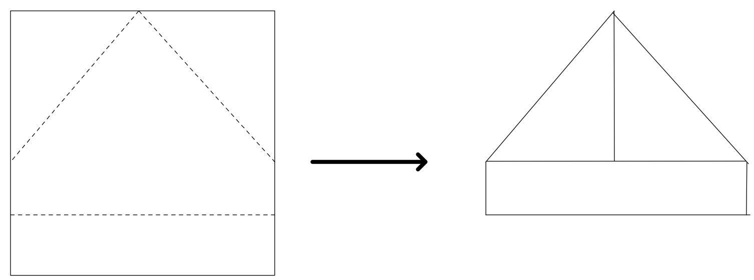

Figure 2: A flat 2-folding of a square to a pentagon

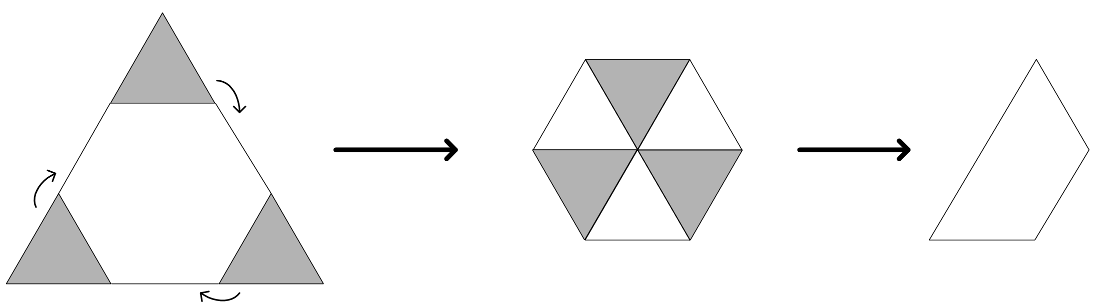

Figure 3: A flat 3-folding of an equilateral triangle to a trapezoid

2 Flat 2-foldings of Regular n-Gons

2.1 The regular pentagon and the regular n-gons, $n \geq 7$

Let P be a regular n-gon. The interior angle at a vertex of P_n is $\Theta_n = \pi - \frac{2\pi}{n}$. In a flat 2-folding of P_n, a necessary condition for a vertex v of P_n to coincide with an interior point

p of P_n is that there exist some positive integer $m \leq n$, such that $m\Theta_n = 2\pi$ or $m\Theta_n = \pi$; see Figure 4. If $m\Theta_n = \pi$, then P_n has an edge that is incident with vertex v in the flat 2-folding. These inequalities can not be satisfied for $n = 5$, or any $n \geq 7$. This proves the following:

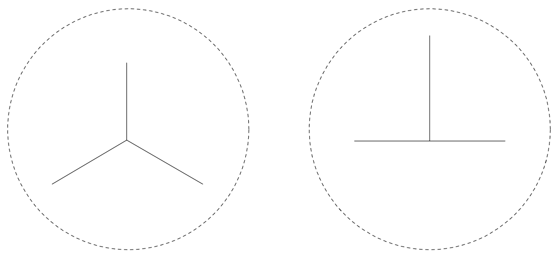

Figure 4: Here $3\Theta_3 = 2\pi$, and $2\Theta_4 = \pi$

Proposition 1 *Any flat 2-folding of a regular $(2n+1)$-gon, $n \geq 2$, can be obtained by folding along one of its lines of symmetry. Flat 2-foldings of a regular $2n$-gons, $n \geq 4$, can be obtained by folding either along a line of symmetry that bisects two opposite sides or one that bisects two opposite angles.*

Figures 5 and 6 provide an illustration of Proposition 1.

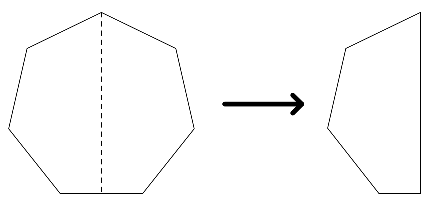

Figure 5: A flat 2-folding of P_7

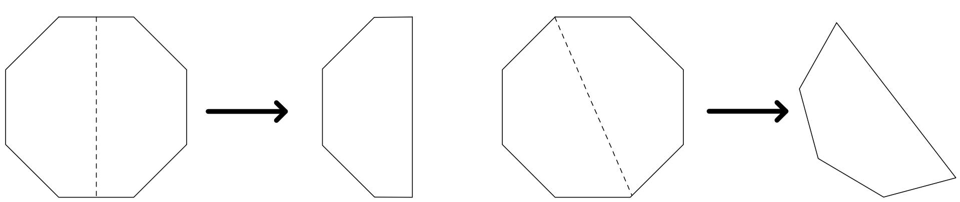

Figure 6: Flat 2-foldings of P_8

2.2 The regular hexagon

One way to obtain a flat 2-folding of a regular hexagon is to fold along one of its lines of symmetry. If any other flat 2-foldings exist, then the necessary condition mentioned earlier must be satisfied. Certainly $3\Theta_6 = 2\pi$, and in fact two other ways to obtain a flat 2-folding can be found:

1. Choose three alternate vertices of the hexagon. The other three vertices determine an equilateral triangle. Fold the hexagon along the sides of this triangle so that the three chosen vertices meet at the center of the hexagon.
2. Choose two adjacent sides of the hexagon and their opposite sides. The midpoints of these four sides determine a rectangle. Fold the hexagon along the sides of this rectangle so that each set of three consecutive vertices enclosing the chosen adjacent sides meets at a point.

It is easy to check that no other flat 2-foldings exist. This proves the following.

Proposition 2 *The foldings shown in Figure 7 are all the possible flat* 2*-foldings of a regular hexagon.*

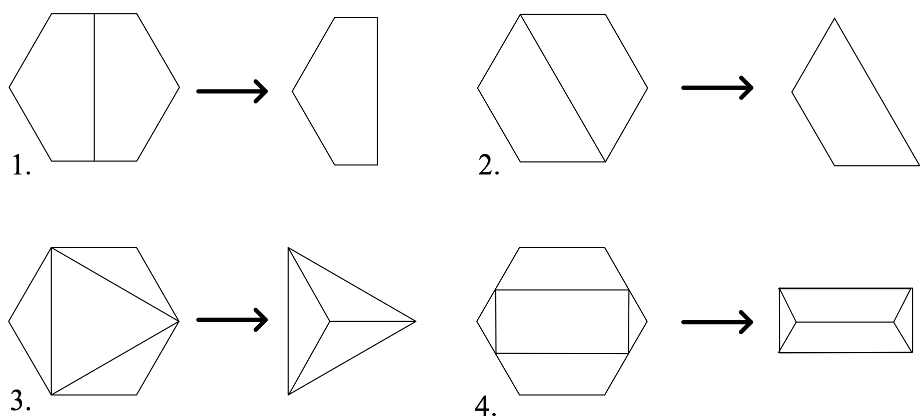

Figure 7: All possible flat 2-foldings of a regular hexagon

2.3 The square

To obtain flat 2-foldings of a square, we can again fold along lines of symmetry. Alternatively, we may note that the necessary condition is satisfied; $4\Theta_4 = 2\pi$ and $2\Theta_4 = \pi$, and search for ways in which the four vertices of the square can coincide with an interior point of the square or two vertices of the square can coincide with an interior point and be incident with a side of the square. If any other flat 2-foldings exist, each vertex of the square must coincide with a point on a side of the square. Such foldings are obtained as follows. Choose two parallel lines in the interior of the square such that the lines are a distance $\frac{1}{2}l$ apart, where l is the length of the side of the square; and fold the square along these lines. These considerations lead to the following proposition.

Proposition 3 *The foldings shown in Figure 8 are all the possible flat* 2*-foldings of a square.*

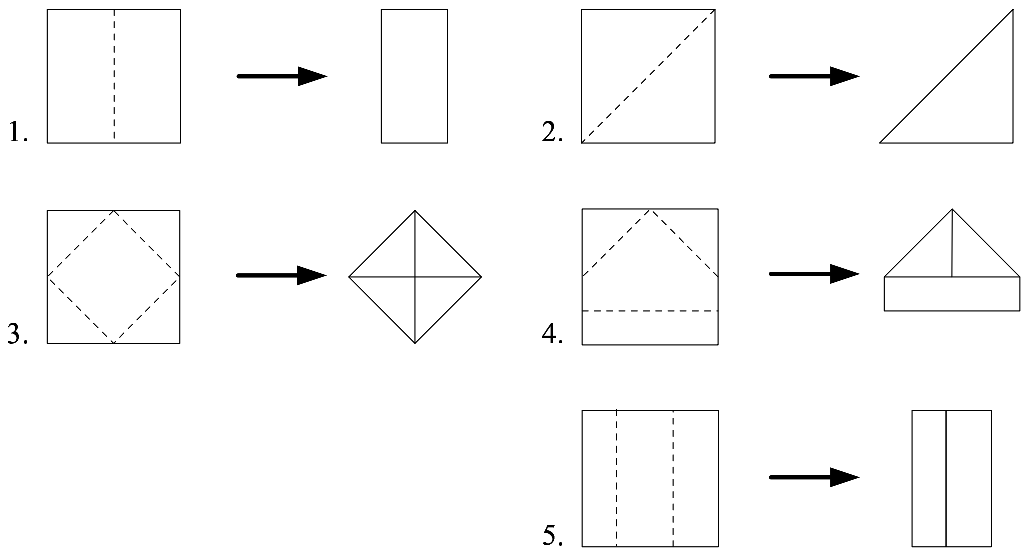

Figure 8: All possible flat 2-foldings of a square

2.4 The equilateral triangle

A flat 2-folding of an equilateral triangle can be obtained by folding along a line of symmetry. Although the necessary condition is satisfied; $3\Theta_3 = \pi$, there is no way that the three vertices of the triangle can meet at an interior point and be incident with a side of the triangle. Hence, if any other flat 2-foldings of the triangle exist, each vertex of the triangle must coincide with a point on a side of the triangle. In fact, this point must be the midpoint of a side; otherwise, a flat 2-folding will not be possible.

If the point is the midpoint of the side opposite the vertex, then the remaining uncovered surface areas will consist of equilateral triangles (see Figure 9). These isosceles triangles can be folded into themselves in three essentially different ways (Figure 10).

If the point is the midpoint of an adjacent side, the result is the configuration shown in Figure 11b). Consider the vertex C. If C is made to coincide with the midpoint M, then the resultant folding is the same as that of Figure 10a). If C is made to coincide with L, then the folding that results is essentially the same as that of Figure 10 b). If C is made to coincide with N, then the folding that results will either be the same as that of Figure 12b) or c), according to whether A is made to coincide with L or N.

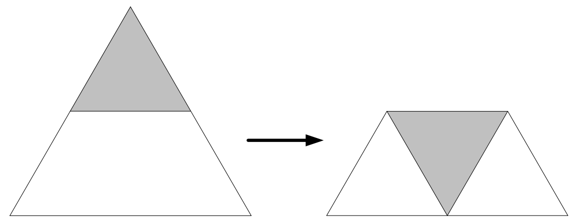

Figure 9:

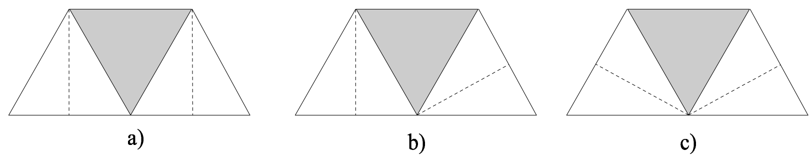

Figure 10:

The possibilities for vertex A can be identified in the same way. They will be included among the foldings shown in Figure 10.

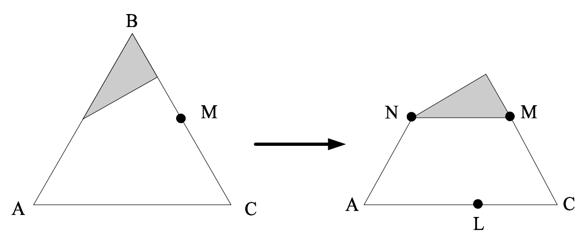

Figure 11:

Hence only four different flat 2-foldings can be obtained from an equilateral triangle. This proves the following proposition.

Proposition 4 *The foldings shown in Figure 12 are all the possible flat 2-foldings of an equilateral triangle.*

3 Convex Polygons Flat 2-foldable to Regular Polygons

In the previous section, we answered the question of what convex polygons can result from flat 2-foldings of regular polygons? In this section, we turn the question around; what convex polygons can be flat 2-folded to regular polygons ?

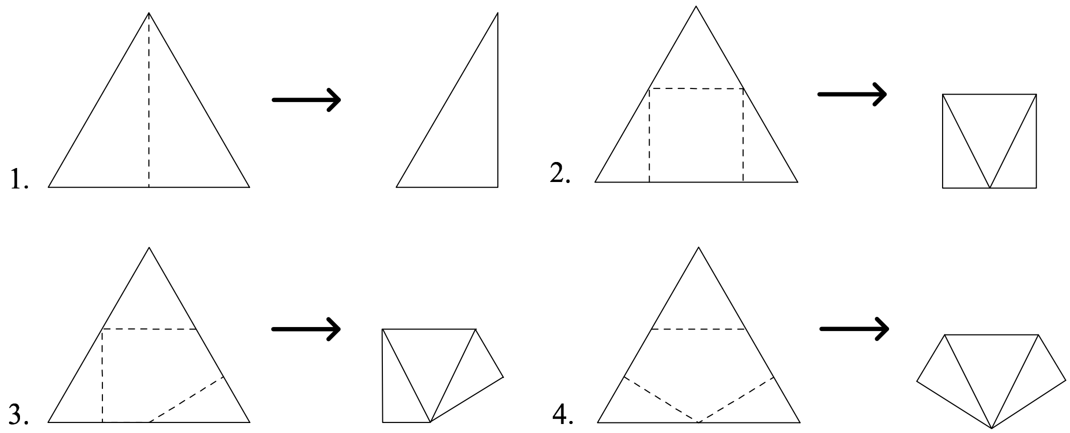

Figure 12:

3.1 Regular Polygons with at Least Five Vertices

We first prove that no convex polygon is flat 2-foldable to a regular n-gon, $n \geq 5$. Let P be a convex polygon with vertices $\{p_1, \ldots, p_n\}$. For each i, let α_i be the internal angle of P at vertex p_i and let $t(p_i) = \pi - \alpha_i$ (see Figure 13).

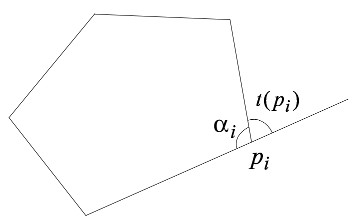

Figure 13:

Observation 1: $\sum_{i=1}^{n} t(p_i) = 2\pi$.

Let Q be a flat 2-folding of P, and $\{q_1, \ldots, q_m\}$ and $\{\beta_1, \ldots, \beta_m\}$ the vertices and angles of Q. Consider an angle $\beta_i > \frac{\pi}{2}$ of Q. Since Q is a flat 2-folding of P, one or more vertices of P are mapped to q_i.

Case 1: Exactly one vertex p_j of P is mapped onto q_i. This case happens when q_i was obtained by folding P along an edge and p_j is mapped to q_i (see figure 14). In this case $t(p_j) = 2t(q_i)$

Case 2: Suppose that $k \geq 2$ vertices $p_1, \ldots, p_k$ of P are mapped to q_i. Observe that

$$\sum_{j=1}^{k} \alpha_j \leq 2\beta_i.$$

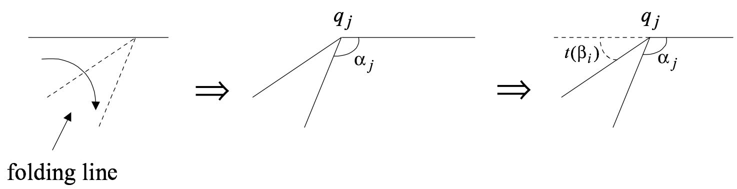

Figure 14:

Since

$$\sum_{j=1}^{k} t(p_j) = \sum_{j=1}^{k} (\pi - \alpha_j) = k\pi - \sum_{j=1}^{k} \alpha_j \geq k\pi - 2\beta_i \geq 2(\pi - \beta_i) = 2t(q_i),$$

we have proved the following.

Lemma 1 *Let Q be a flat 2-folding of P and q_i a vertex of Q such that $\beta_i > \frac{\pi}{2}$. Then if $p_1, \ldots, p_k$ are mapped to q_i,*

$$\sum_{j=1}^{k} t(p_j) \geq 2t(q_i).$$

Theorem 1 *No convex polygon is flat 2-foldable to a regular n-gon, $n \geq 5$.*

Proof: Let Q_n be a regular n-gon, $n \geq 5$. Then all internal angles of Q_n are greater than $\frac{\pi}{2}$. Observe that

$$\sum_{i=1}^{n} t(q_i) = 2\pi.$$

Suppose that Q_n is a flat 2-folding of some convex polygon P. For each i, let S_i be the set of vertices of P mapped in the folding to q_i, $1 \leq i \leq n$. Observe that $S_i \cap S_j = \emptyset$, $i \neq j$. By Lemma 1,

$$t(S_i) = \sum_{p_j \in S_i} t(p_j) \geq 2t(q_i),$$

and thus

$$\sum_{i=1}^{n} t(S_i) \geq 2 \sum_{i=1}^{n} t(q_i) = 4\pi,$$

which is a contradiction. ∎

In view of the theorem, we now proceed to study the set of convex polygons flat 2-foldable to equilateral triangles and squares.

3.2 The equilateral triangle

A way to obtain convex polygons flat 2-foldable to an equilateral triangle is as follows. Take an equilateral triangle T and a point p on it. For each edge e_i of T, let p_i be the mirror image of p with respect to the line generated by e_i (see Figure 15).

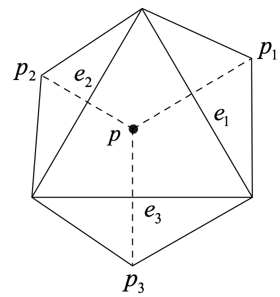

Figure 15:

Let P be the polygon whose vertices are p_1, p_2, p_3 and the vertices of T. Depending on the position of p, we obtain a hexagon (Figure 15), a pentagon, or a quadrilateral (Figure 16). It is now not difficult to see that with this procedure, all polygons flat 2-foldable to an equilateral triangle will be generated.

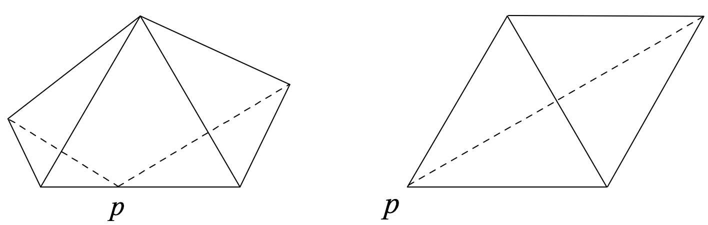

Figure 16:

3.3 The square

There are two types of polygons which are flat 2-foldable to a square; those that contain a copy of S (which is not folded in the flat 2-folding of P), and those which do not contain such a copy of S; see Figure 17. In this paper we confine ourselves to foldings of the first type and characterize them. Foldings of the second type will be characterized in a forthcoming paper.

Let Q be a polygon and P a convex polygon flat 2-foldable to Q. Consider a flat 2-folding of P to Q. A point q of Q, *not* a vertex of Q, is called *singular* if in the folding of P to Q, at least one vertex of P is mapped to q. There are two types of singular points, those lying in

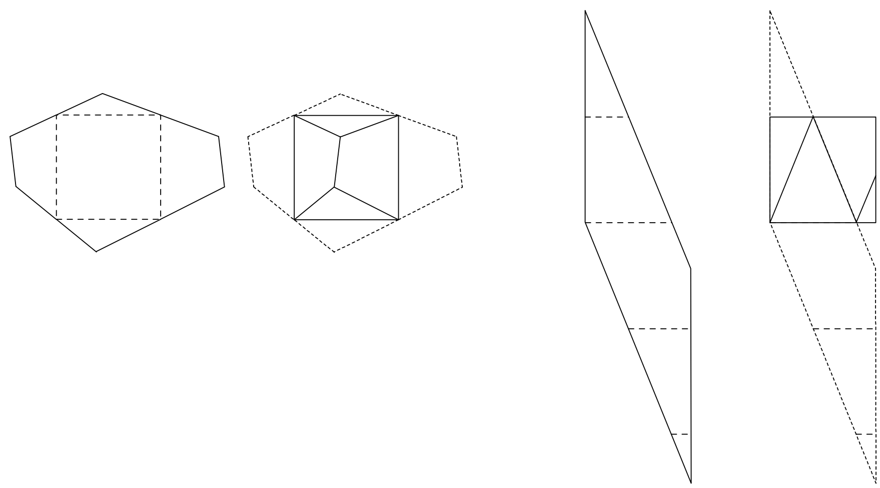

Figure 17:

the interior of Q, and those lying in the relative interior of edges of Q. The following three lemmas are given without proof.

Lemma 2 *Let S_i be the set of vertices of P mapped to a singular point q of Q. Then*

$$\sum_{p_i \in S_i} t(p_i) \geq \pi.$$

Lemma 3 *In a flat 2-folding of P to Q, Q has at most two singular points.*

Let q be a singular point of Q such that k vertices of P are mapped to Q. We call k the degree of q.

Lemma 4 *The degree of any singular point of Q is at most four. Moreover if Q has a singular point q of degree four, then q is the only singular point of Q.*

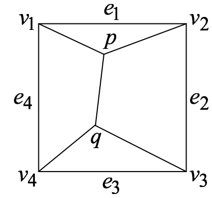

Figure 18:

Consider the labeling on the vertices and edges of S as in Figure 18, and p, q, two points on S, *not necessarily different*. Suppose that we join p to v_1, v_2 and q, and q to v_3 and v_4 using non-crossing line segments. We also require that the angles formed around p and q by these line segments be less that or equal to π, as shown in Figure 18. Let p_1, p_2 and p_4 be the mirror images of p with respect to e_1, e_2, and e_4. Define q_2, q_3, and q_4 similarly; see Figure 19.

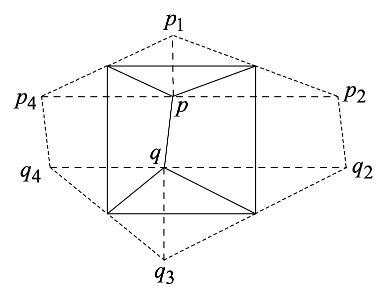

Figure 19:

Clearly the polygon with vertex set $\{p_1, p_2, q_2, q_3, q_4, p_4\}$ is flat 2-foldable to S. According to the position of p and q, we obtain the cases shown in Figure 20, which characterize the set of all polygons of the first type flat 2-foldable to a square. It is now not difficult to show, using the preceding lemmas, that any convex polygon flat 2-foldable to S can be obtained from one of the cases shown in Figure 20.

4 Further research

The problem of determining all flat n-foldable convex polygons $n \geq 2$ remains open. In a forthcoming paper, we identify all convex polygons that are flat 2-foldable to a square, and those that are flat 3-foldable to a triangle.

A more specific question is the following. Are there any convex polygons other than the rectangle which are flat n-foldable for any n?

References

[1] E. Demaine, M. Demaine, A. Lubiw and J. O'Rourke, Enumerating foldings between polygons and polytopes, *Graphs and Combinatorics* 18(1)2002, 93-104.

[2] J. Akiyama and G. Nakamura, Foldings of Regular Polygons to Convex Polyhedra I: Equilateral Triangles, to appear in *Proc. IJCCGT*, Springer Verlag LNCS (2004).

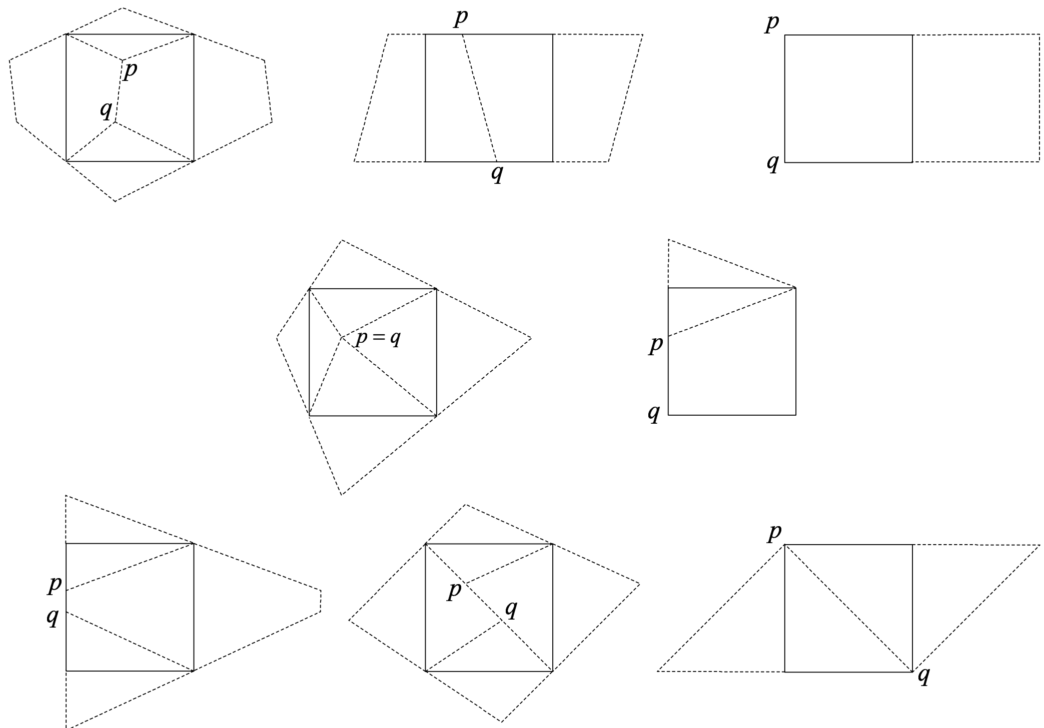

Figure 20:

[3] J. Akiyama and G. Nakamura, Foldings of Regular Polygons to Convex Polyhedra II: Regular Pentagons, *J. Indonesia Math. Soc.* Vol.9(2003)pp.89-99.

[4] J. Akiyama and G. Nakamura, Foldings of Regular Polygons to Convex Polyhedra III: Regular Hexagons and Regular n-Gons, $n \geq 7$, *J. Thai Math. Soc.*

[5] R. Alexander, H. Dyson and J. O'Rourke, The foldings of a square to convex polyhedra, *Discrete and Computational Geometry* (JCDCG 2002), Springer Verlag LNCS 2866, pp.38-50 (2003).