Continuous surveillance of points by rotating floodlights

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Abstract

Let P and F be sets of $n \geq 2$ and $m \geq 2$ points in a plane, respectively. We study the problem of finding the minimum angle $\alpha \in [2\pi/m, 2\pi]$ such that one can install at each point of F a stationary rotating floodlight with illumination angle α, initially oriented in a suitable direction, in such a way that, at all times, every target point of P is illuminated by at least one floodlight. All floodlights rotate clockwise at unit speed. We provide bounds for the case in which the elements of $P \cup F$ are on a given line, and present exact results for the case in the plane in which we have two floodlights and many target points. We further consider the non-rotating version of the problem and look for the minimum angle α such that one can install a non-rotating floodlight with illumination angle α at each point of F, in such a way that every target point of P is illuminated by at least one floodlight. We show that this problem is NP-hard and hard to approximate.

Keywords: Floodlight scheduling; illumination; rotating sensors; coverage.

1 Introduction

Art gallery theory and illumination problems are well-known in Discrete and Computational Geometry. There are many variations on this area which are surveyed by Urrutia[12] and O'Rourke[8].

An important and practical issue that arises in the automation of various security, surveillance, and reconnaissance tasks is that of observing continuously a set of targets in an area of interest. A key issue in these problems is the placement of the floodlights (i.e. sensors), that is, determining where to locate a set of static (i.e. non-rotating) floodlights to maintain the targets under view. In this static model, the number of floodlights is fixed in advance to ensure adequate illumination of the area of interest. We define a *floodlight* as a point in the plane, and its *illumination region* as a wedge with apex at the point, that is, a region bounded by two halflines emanating from the point.

The scheduling of static floodlights for covering a given region was first considered by Bose et al.[1]: Given m points in the plane, and m angles $\alpha_1, \alpha_2, \ldots, \alpha_m$, where each α_i is at most π,

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floodlights of the given angles can be placed at the given points so as to illuminate the whole plane if and only if $\alpha_1 + \cdots + \alpha_m \geq 2\pi$. Furthermore, for the case where $\alpha_1 + \cdots + \alpha_m \geq 2\pi$, they provided an $O(m \log m)$-time algorithm to place the floodlights.

In practical situations, the area may be so large that economical concerns prohibit the placement of the required number of static floodlights. In these circumstances, the available floodlights will be insufficient to cover the terrain of interest. This suggests the possibility of using multiple floodlights that continuously rotate over time, perhaps with a wider illumination angle to compensate for the reduction in the number of floodlights. This is the variant of the problem introduced by Kranakis et al.[6]: Given the locations of the rotating floodlights (or in their case, antennae) and a region, their problem asks for scheduling the floodlights so that the *entire* region is covered at all times. An important advantage in considering rotating floodlights is that in addition of covering the target region at all times, we are able to monitor all other points of the plane periodically. This increases the degree of fault-tolerance of the entire system as even if one or more floodlights fail, every target point is illuminated part of the time. If, on the contrary, stationary floodlights are used, then there might be several regions of the plane that are never illuminated.

In this article, we investigate a discrete version of the problem studied by Kranakis et al., namely, the problem of continuously covering a discrete set of points. We define the *Rotating Floodlights Illumination Problem* as follows. Let P be a set of $n \geq 2$ points in the plane called *target points*, or simple targets, and F be a set of $m \geq 2$ other points in the plane representing the location of rotating floodlights. We assume that all the floodlights rotate clockwise at unit speed. We say that *F covers P with (illumination) angle α* if there exists a suitable initial orientation of each floodlight with angle α so that, at all times, each target point of P is illuminated by at least one floodlight. When α is known, we simple say that F covers P. Obviously, if the illumination angle is smaller than $2\pi/m$, then every target point of P will not be covered at some time, independently of how the floodlights are oriented initially. We consider the problem of finding the minimum angle $\alpha = \alpha(P, F) \in [2\pi/m, 2\pi]$, and the initial orientation of each floodlight, so that F covers P. We emphasize that the discrete version of the problem considered here is more complicated than the continuous one. For instance, while there is a simple formula for illuminating a given line by a finite set of rotating floodlights on that line, the continuous illumination of discrete target point sets in the line can sometimes be realized with smaller angles by using a better strategy depending on the relative locations of the target points and the floodlights.

We should point out that our problems are closely related to the problems studied by Fusco and Gupta[5]. They consider a finite set of target points and a finite set of directional sensors, both in the plane. Each sensor has a closed sensing region (i.e. illumination region) and a finite number of possible orientations. The time is discretized and at every instance of time each sensor changes from its current orientation to the next one. The *dark time* of a point is the number of time slots during which the point is not covered by any sensor. They then look for the initial orientation of the sensors such that a certain function of the dark time of the target points is minimized. They consider that both the possible orientations and the (bounded) illumination region of each sensor are arbitrary, and prove that deciding the possibility of zero dark time, i.e., whether these exists an initial orientation of the sensors such that at every discrete instance of time every target point is illuminated, is NP-complete. They mention, without any explicit proof, that deciding zero dark time is NP-complete even if the illumination regions of the sensors are cone-shaped. Since a cone-shaped region is not precisely a wedge, and the set of possible orientations of each antenna is arbitrary, this hardness result does not imply that our problem is NP-hard in the more general setting.

In addition to the above rotating floodlights problems, we study the problem of finding the minimum angle α such that a non-rotating floodlight of angle α can be installed at each point of F so that the floodlights illuminate all the points of P.

Further research related to our problems can be found in many domains, including art gallery and related problems, multi-target tracking, placement and orientation of rotating directional sensors, and multi-robot surveillance tasks[2, 3, 4, 5, 9, 10, 11, 12].

1.1 Results

In this paper we study restricted instances of the *Rotating Floodlights Illumination Problem*, leaving the general solution as an interesting open problem. Concretely, we provide results for the following two cases: (i) the scenario where the elements of $P \cup F$ are located on a given line (Section 2), and (ii) the planar version with two floodlights and an arbitrary number of targets (Section 3). For case (i), where the elements of $P \cup F$ are on a line (say the x-axis), suppose that $P_1, P_2, \ldots, P_{k-1}$ is the minimum-cardinality partition of P such that each P_i is formed by consecutive elements in $P \cup F$. Let m_1 denote the number of elements of F to the left of the elements of P_1, m_i denote the number of elements of F to the right of the elements of P_i and to the left of the elements of P_{i+1} for $i = 2, \ldots, k-1$, and m_k denote the number of elements of F to the right of the elements of P_{k-1}. Let Q be the number of odd numbers in the sequence $\{m_1 + m_k, m_2, \ldots, m_{k-1}\}$. If all the numbers $m_1 + m_k, m_2, \ldots, m_{k-1}$ are zero, except one of them that is equal to m, we show that the angle $\alpha = 2\pi/m$ is optimal. Otherwise, we show the following bounds:

$$\frac{2\pi}{m - \frac{Q}{3}} \leq \alpha(P, F) \leq \min\left\{\frac{3\pi}{m}, \frac{2\pi}{m - Q + 2\lfloor\frac{Q}{3}\rfloor}\right\}.$$

In case (ii) we consider two situations. In the first one, there are elements of P on both sides of the line through the two floodlights f_1 and f_2. Then, $\alpha(P, F) = \pi + (\beta^+ + \beta^-)/2$, where β^+ is the maximum of the angles $\angle f_1 p f_2$ of the points $p \in P$ in one side of that line, and β^- is the maximum of the angles $\angle f_1 p' f_2$ of the points $p' \in P$ in the other side. In the second one, all the points of P are on the same side of the line through f_1 and f_2. Here, $\alpha(P, F) = \pi + (\beta_{max} - \beta_{min})/2$, where β_{max} and β_{min} are the maximum and the minimum, respectively, of the angles $\angle f_1 p f_2$ of the points $p \in P$. In both cases (i) and (ii), our proofs lead to simple algorithms for determining the initial orientation of the rotating floodlights. The problem of non-rotating floodlights is studied in Section 4. We show that this problem is NP-hard and hard to approximate.

1.2 Notation

Given points u, v in the plane, $\ell(u, v)$ denotes the line containing both u and v. Given a floodlight $f \in F$ and an angle $\beta < 2\pi$, we say that we configure f *with angle* β if f rotated (clockwise) with angle β starts to illuminate the positive x-axis. That is, f illuminates the positive x-axis if f is rotated with angle β, whereas f rotated with angle $\beta - \varepsilon$ does not for all arbitrary small enough positive values of ε. Given that all floodlights rotate at the same speed, it suffices to consider only the interval of time $[0, 2\pi)$.

2 Points and floodlights on a line

We first consider the case in which the points of P and the floodlights of F lie on a line $\mathcal{L}$, say the x-axis. Kranakis et al.[6] considered the case in which the floodlights are located on a line that they need to illuminate. They showed that the whole line can be illuminated by m rotating floodlights using illumination angle $3\pi/m$ and that this bound is tight. This can be viewed as a special case of our problem in which $n \geq m+1$ and each of the $m+1$ segments of the line determined by F contains at least one point of P. We consider other cases and show that the illumination angle is smaller than $3\pi/m$ for some of them.

We partition P into $k-1$ ($k \geq 2$) maximal intervals denoted $s_1, s_2, \ldots, s_{k-1}$ from left to right, each of which contains elements of P but no elements of F. Note that two points of P belong to the same subset if and only if there exists no floodlight between them. Let F_1 denote the elements of F to the left of s_1, F_i ($i = 2, \ldots, k-1$) denote the elements of F between s_{i-1} and s_i, and F_k denote the elements of F to the right of s_{k-1}. Let $m_i = |F_i|$ for $i = 1, \ldots, k$. Observe that $m_1, m_k \geq 0$, $m_i \geq 1$ for $i = 2, \ldots k-1$, and $m_1 + m_2 + \ldots + m_k = m$.

Lemma 1 *Two floodlights f_1 and f_2 with illumination angle $\alpha \leq \pi$, belonging to the same set among $F_1 \cup F_k, F_2, F_3, \ldots, F_{k-1}$, can be configured to illuminate P during one or two intervals of total length 2α. Furthermore, if two floodlights of F cover P (at all times) with angle $\alpha < 3\pi/2$, then they must belong to the same set among $F_1 \cup F_k, F_2, F_3, \ldots, F_{k-1}$.*

Proof. Suppose that both f_1 and f_2 belong to the set F_i ($i = 1, \ldots, k$), and assume w.l.o.g. that f_1 is to the left of f_2. Configure f_1 with angle zero and f_2 with angle π (see the top of Figure 1a). At time $t = \pi$ the configuration of f_1 and f_2 is as shown in the bottom of Figure 1a. Since there is no element of P in the segment connecting f_1 and f_2 then all elements of P are illuminated during the time intervals $[0, \alpha]$ and $[\pi, \pi + \alpha]$, for 2α time in total.

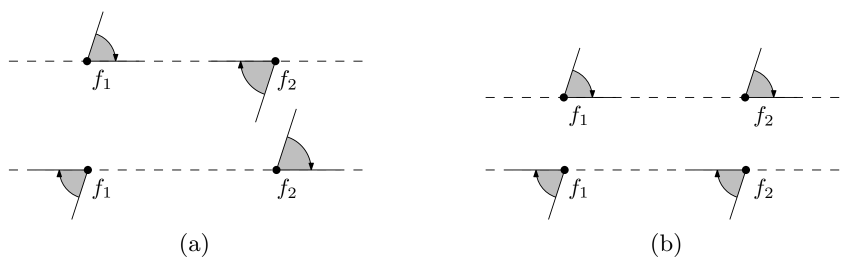


Figure 1: Proof of Lemma 1: (a) Case where f_1 and f_2 belong to the same set F_i. (b) Case where $f_1 \in F_1$ and $f_2 \in F_k$.

Suppose now that $f_1 \in F_1$ and $f_2 \in F_k$. Configure both f_1 and f_2 with angle zero (see the top of Figure 1b). At time $t = \pi$ the configuration of f_1 and f_2 is as shown in the bottom of Figure 1b. Since all elements of P belong to the segment connecting f_1 and f_2 then all elements of P are illuminated during the intervals $[0, \alpha]$ and $[\pi, \pi + \alpha]$, 2α time in total.

For the second part of the lemma, let f and g be the two floodlights that cover P. Clearly $\alpha \geq \pi$. Assume, w.l.o.g., that $g \notin F_1 \cup F_k$. Then, it is trivial to see that f and g must belong to the same set whenever $\alpha \geq 3\pi/2$. □

Lemma 2 *Three floodlights f_1, f_2, and f_3 with illumination angle $\alpha < \pi$, can be configured so that, together, they cover the whole line $\mathcal{L}$ (hence, P) during 2α time in total.*

Proof. The proof can be obtained from the arguments of Kranakis et al.[6]. Assume, w.l.o.g., that f_1, f_2, and f_3 appear in this order from left to right. Configure f_1, f_2, and f_3 with angle zero, π, and zero, respectively (top of Figure 2). At time $t = \pi$ the configuration is as shown at the bottom of Figure 2. During the interval $[0, \alpha]$ of time the line is illuminated by f_1 and f_2, and during the interval $[\pi, \pi + \alpha]$ by f_2 and f_3. The result follows. □

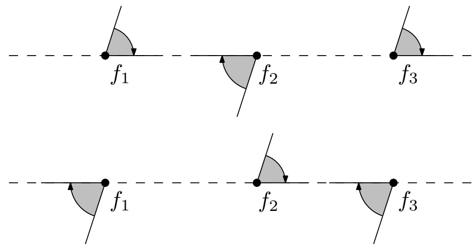


Figure 2: Proof of Lemma 2

We now establish the main result for the case in which the target points and the floodlights lie on a common line.

Theorem 1 *If all floodlights belong to the same set among $F_1 \cup F_k$,F_2,$F_3, \ldots, F_{k-1}$, then $\alpha(P, F) = 2\pi/m$, which is optimal. Otherwise, $\alpha(P, F)$ satisfies:*

$$\frac{2\pi}{m - \frac{Q}{3}} \leq \alpha(P, F) \leq \min\left\{\frac{3\pi}{m}, \frac{2\pi}{m - Q + 2\lfloor \frac{Q}{3} \rfloor}\right\} \tag{1}$$

where Q denotes the number of odd numbers in the set $\{m_1 + m_k, m_2, \ldots, m_{k-1}\}$.

Proof. Obviously $\alpha(P, F) \geq 2\pi/m$ in all cases. All the floodlights belong to the same set among $F_1 \cup F_k$,F_2,$F_3, \ldots, F_{k-1}$ if and only if $k = 2$, or $k = 3$ and $m_1 = m_3 = 0$. In both cases, the illumination angle $\alpha = 2\pi/m$ is sufficient. Assume first that $k = 2$ and let $F_1 = \{f_1, \ldots, f_{m_1}\}$ and $F_2 = \{f'_1, \ldots, f'_{m_2}\}$. Floodlight f_i is configured with angle $(i-1)\alpha$ for $i = 1, \ldots, m_1$, and floodlight f'_j, with angle $\pi - j\alpha$ for $j = 1, \ldots, m_2$ (see Figure 3). Then, at any time $t \in [0, m_1\alpha)$ P is covered

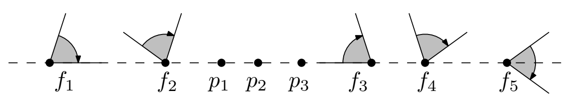


Figure 3: Two groups of floodlights $F_1 = \{f_1, f_2\}$ and $F_2 = \{f'_1, f'_2, f'_3\} = \{f_3, f_4, f_5\}$, where $m_1 = 2$ and $m_2 = 3$, and a configuration with angle $\alpha = 2\pi/5$.

by an element of F_1 and, at any time $t \in [m_1\alpha, 2\pi)$, by an element of F_2. Assume now that $k = 3$ and $m_1 = m_3 = 0$. Then, $F = F_2 = \{f_1, \ldots, f_m\}$. By configuring f_i with angle $2i\pi/m$, P is covered by F. This proves optimality when all the floodlights belong to the same set F_2,$F_3, \ldots, F_{k-1}$, or $F_1 \cup F_k$.

Assume now that neither of the two cases above occurs. We first show that the upper bound in Equation 1 is sufficient by configuring the floodlights as follows. We consider the case $m_1 = m_k = 0$ and $m_i = 1$ $(i = 2, \ldots, k-1)$ separately because the result follows immediately from Kranakis et al.[6], since our problem is equivalent to illuminating the whole x-axis. In this case, $m = Q$ and $\alpha(P, F) = 3\pi/m = \min\left\{\frac{3\pi}{m}, \frac{2\pi}{m-Q+2\lfloor \frac{Q}{3} \rfloor}\right\}$, which is also optimal.

For the remaining cases, we proceed as follows. Pair the elements of F into $N = \lfloor \frac{m_1+m_k}{2} \rfloor + \lfloor \frac{m_2}{2} \rfloor + \ldots + \lfloor \frac{m_{k-1}}{2} \rfloor = \frac{m-Q}{2}$ pairs: $(f_{1,1}, f_{1,2})$, $(f_{2,1}, f_{2,2})$, $\ldots$, $(f_{N,1}, f_{N,2})$ so that the elements of each pair belong to the same set among $F_1 \cup F_k, F_2, F_3, \ldots, F_{k-1}$. Group the remaining Q floodlights into $M = \lfloor \frac{Q}{3} \rfloor$ triples $(f'_{1,1}, f'_{1,2}, f'_{1,3}), \ldots, (f'_{M,1}, f'_{M,2}, f'_{M,3})$ (leaving at most two elements of Q ungrouped). Let $\alpha = \frac{2\pi}{m-Q+2\lfloor \frac{Q}{3} \rfloor} = \frac{2\pi}{2N+2M}$. We now schedule the floodlights as follows. Configure

$f_{i,1}$ and $f_{i,2}$ with angles $(i-1)\alpha$ and $\pi + (i-1)\alpha$, respectively, for $i = 1, \ldots, N$; and configure $f'_{j,1}$, $f'_{j,2}$, and $f'_{j,3}$ with angles $(N+j-1)\alpha$, $\pi + (N+j-1)\alpha$, and $(N+j-1)\alpha$, respectively, for $j = 1, \ldots, M$. Finally, arbitrarily configure the remaining floodlights (at most two). The correctness of this configuration follows from lemmas 1 and 2.

To establish the lower bound in Equation 1, we use the geometric interpretation of *dual space* proposed by Kranakis et al. [6]. Let $\alpha = \alpha(P, F)$ be the optimal angle and let z_0 denote a suitable initial configuration of the floodlights, with illumination angle α, at time $t = 0$. Furthermore, let C represent the unit circle with center o and let ℓ be a directed line through o oriented horizontally and pointing in the positive x-axis direction. Translate each floodlight $f \in F$ to the center o, keeping the same orientation as in z_0. Then, each floodlight $f \in F$ maps to the circular arc c_f of C of amplitude α (called an *f-arc*) and the (clockwise) rotation of the floodlights maps to the counter-clockwise rotation of the line ℓ, around o, keeping static the f-arcs. Indeed, at any instant of time $t \in [0, 2\pi)$, a floodlight f illuminates the positive (resp. negative) direction of the line $\mathcal{L}$ if and only if the line ℓ, rotated with angle t, intersects the static arc c_f and the oriented segment from the center o to the point $\ell \cap c_f$ points in the same (resp. contrary) direction as ℓ.

Observe from Lemma 1 and Lemma 2 that at any instant of time (i.e. any position of ℓ) the line ℓ intersects at least two f-arcs. Let T_2 be the set of instants in the interval $[0, 2\pi)$ such that the line ℓ intersects exactly two f-arcs, and let $T_{\geq 3} = [0, 2\pi) \setminus T_2$ denote the complement of T_2. Note that both T_2 and $T_{\geq 3}$ are the union of pairwise disjoint arcs of C. Let a and b denote the total amplitude of the arcs in T_2 and $T_{\geq 3}$, respectively.

Obviously, $a + b = 2\pi$. We also have that $m\alpha \geq a + (3/2)b$ since at every instance of time of T_2 the line ℓ intersects exactly two f-arcs, and at every instance of time of $T_{\geq 3}$ it intersects at least three f-arcs. Hence $3\pi - \alpha m \leq a/2$. We further have that T_2 has at most N pairs of floodlights and, by Lemma 1, each pair can cover P during at most 2α time, accordingly $a \leq 2N\alpha$. We then obtain that $3\pi - \alpha m \leq N\alpha$, which implies that $\frac{2\pi}{m - \frac{Q}{3}} \leq \alpha$, as claimed. □

3 Many points and two lights

In this section, we consider the case of two floodlights f_1 and f_2, i.e., $m = 2$. Let $p_1, \ldots, p_n$ denote the elements of P. Assume w.l.o.g. that the line $\ell(f_1, f_2)$ is horizontal and that f_1 is located to the left of f_2. Given any target point p_i ($i = 1, \ldots, n$), let $\theta_i \in [0, \pi)$ denote the angle at p_i in the triangle $\triangle p_i f_1 f_2$ with vertices p, f_1, and f_2. If there are points from P on both sides of the line $\ell(f_1, f_2)$, then we define two angles β^+ and β^- as the maximum of θ_i over all points p_i above and below $\ell(f_1, f_2)$, respectively (see Figure 4a). Otherwise, if all the points of P are on the same side of $\ell(f_1, f_2)$, we define two angles, β_{max} and β_{min}, as the largest and smallest θ_i over all points p_i, respectively (see Figure 4b).

Theorem 2 (Two floodlights) *For $m = 2$, $n \geq 2$:*
(1) If there are points of P on both sides of $\ell(f_1, f_2)$ then $\alpha(P, F) = \pi + \frac{\beta^+ + \beta^-}{2}$.
(2) If all the points of P lie on one side of $\ell(f_1, f_2)$ then $\alpha(P, F) = \pi + \frac{\beta_{max} - \beta_{min}}{2}$.

Proof. First, we prove part (1) of the theorem. We configure the floodlights f_1 and f_2 initially as follows. Let Af_1Bf_2 be the quadrilateral with vertices A, f_1, B, and f_2 such that the angle $\angle f_1 A f_2$ is equal to β^+, the angle $\angle f_2 B f_1$ is equal to β^-, and the points f_1 and f_2 are symmetric with respect to the line $\ell(A, B)$, as shown in Figure 5a.

Since f_1 and f_2 rotate at unit speed, it always holds that $\angle f_1 A f_2 = \beta^+$ and $\angle f_2 B f_1 = \beta^-$ (see Figure 5b). Furthermore, the region not illuminated by the floodlights is always a subset of the

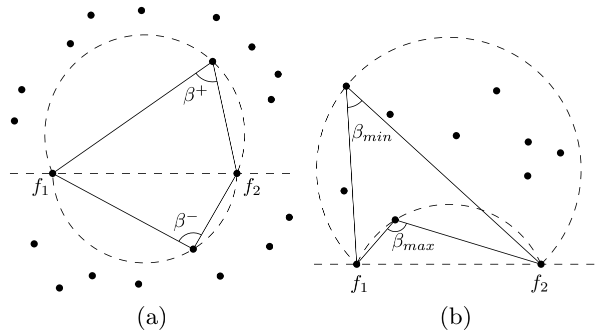


Figure 4: (a) β^+ and β^-. (b) β_{max} and β_{min}.

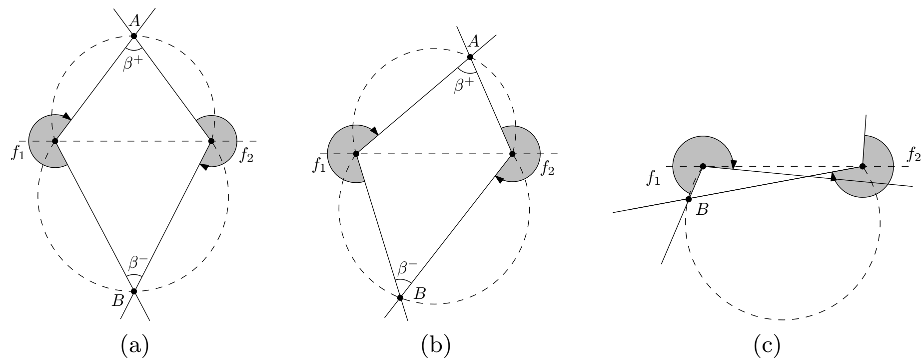


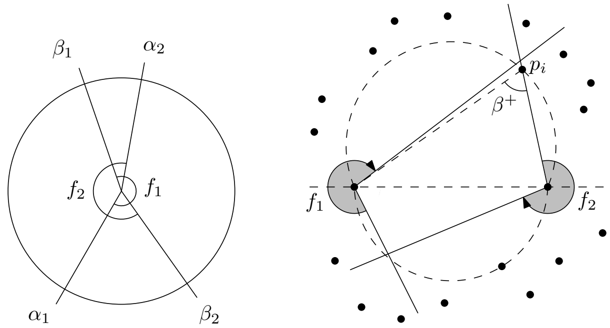
Figure 5: (a) Initial position. (b),(c) General position.

interior of the union of the triangles $\triangle f_1Af_2$ and $\triangle f_2Bf_1$ (see Figures 5b and 5c), and it never contains points of P by the definition of β^+ and β^-. It remains to show that any illumination angle smaller than $\pi + \frac{\beta^+ + \beta^-}{2}$ is not feasible.

Suppose that, initially, floodlight f_i ($i = 1, 2$) covers the directions in the interval $[\alpha_i, \beta_i]$. First, we show that these intervals together cover all the possible directions in $[0, 2\pi]$. If, to the contrary, a direction t is not covered by $[\alpha_1, \beta_1] \cup [\alpha_2, \beta_2]$, then there exists a rotation such that p_i is not illuminated, where p_i is the point above the line $\ell(f_1, f_2)$ and $\angle f_2 p_i f_1 = \beta^+$, a contradiction. Therefore, the floodlight intervals overlap as shown in Figure 6a.

We show now that the overlapping interval $[\alpha_2, \beta_1]$ has length at least β^+. Suppose, to the contrary, that its length is smaller than β^+. Consider the rotation by angle γ such that the β_2-ray of floodlight f_2 passes through point p_i defining β^+ (see Figure 6b). Then the α_1-ray of floodlight f_1 will not cover p_i because the angle between the two rays is less than β^+. Therefore, p_i is not covered if the rotation angle is slightly smaller than γ, a contradiction. Similarly, the overlapping interval $[\alpha_1, \beta_2]$ has length at least β^-. If the illumination angle is α then $2\alpha \geq 2\pi + \beta^+ + \beta_i$. The claim of part (1) follows.

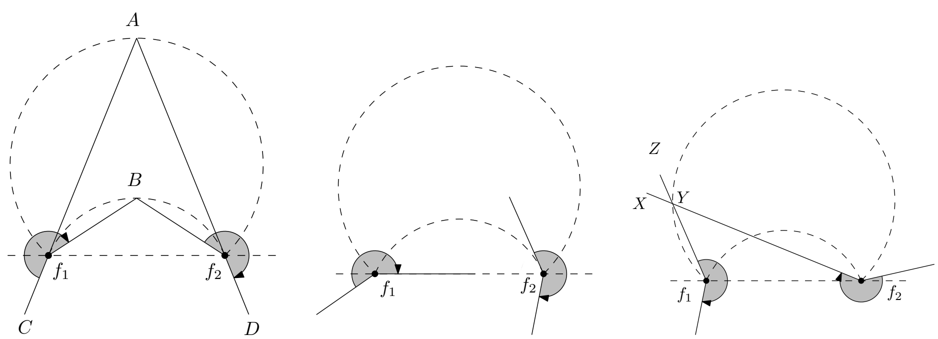
To prove part (2) of the theorem, we configure the floodlights f_1 and f_2 at the beginning as follows. Let Af_1Bf_2 be the quadrilateral such that $\angle f_1Af_2 = \beta_{min}$ and $\angle f_2Bf_1 = \beta_{max}$ and the points f_1 and f_2 appear symmetrically with respect to the line $\ell(A, B)$, as shown in Figure 7a. The argument is similar to the proof of part (1) since points A and B move along the arcs shown in Figure 7a. The uncovered part is either the region Cf_1Bf_2D below the lines $\ell(f_1, A)$, $\ell(f_2, A)$, $\ell(f_1, B)$, and $\ell(f_2, B)$, shown in Figure 7a, or the convex wedge XYZ with apex at point Y, shown in Figure 7c. In any case, the area between the two arcs defined by β_{min} and β_{max} is always



(a) (b)

Figure 6: The floodlight intervals.

illuminated.



(a) (b) (c)

Figure 7: The floodlight intervals.

The optimality of angle $\alpha = \pi + \frac{\beta_{max} - \beta_{min}}{2}$ can be shown similarly to the proof of part (1). $\square$

4 The static illumination problem

In this section, we consider the following canonical problem, which we call the *Minimum Angle Illumination Problem*: Given a set P of n target points and a set F of m (non-rotating) floodlights in the plane, find the minimum angle α such that there exists an orientation of the floodlights with illumination angle α that illuminates all of P. We show that this problem is NP-hard, by showing that deciding whether $\alpha = 0$ is NP-complete. Observe that illuminating all targets with angle $\alpha = 0$ is related to covering all the elements of P with (a minimum number of) lines. This problem is proved to be NP-hard by Megiddo and Tamir [7] using a reduction from the *3SAT Problem*. We extend their construction to prove that the decision version of the *Minimum Angle Illumination Problem* is NP-complete.

Theorem 3 *Deciding whether $\alpha = 0$ in the Minimum Angle Illumination Problem is NP-complete.*

Proof. Given an orientation of the floodlights with angle $\alpha = 0$, one can verify in polynomial time whether all targets are illuminated or not. Hence, this decision question is in NP. To show that it is NP-hard, we will use a reduction from the *3SAT Problem*. Let the N variables $v_1, v_2, \ldots, v_N$ and the M clauses $C_1, C_2, \ldots, C_M$ be an instance of the *3SAT Problem*. Assume that every variable appears at most once in each clause. We start with the construction of Megiddo and Tamir [7], in which $M + N \cdot M^2$ points, M corresponding to the clauses $C_1, \ldots, C_M$ and M^2 points corresponding to each pair of literals $(v_i, \overline{v_i})$, and $2N \cdot M$ lines are constructed. Such a construction can be done in polynomial time and with polynomially-bounded coordinates, and satisfies the following properties (refer to Figure 8):

- Each clause C_j is represented by a point, with a slight abuse of notation, denoted also by C_j.
- Each pair of literals $(v_i, \overline{v_i})$ is represented by a grid G_i of M^2 points, obtained from the intersection points of the M lines $L_{i1}, \ldots, L_{iM}$ with the M lines $\overline{L}_{i1}, \ldots, \overline{L}_{iM}$.
- With the exception of the lines L_{ij} and $\overline{L}_{ij}$, no other line contains more than two points of the set $\{C_1, \ldots, C_M\} \cup G_1 \cup \cdots \cup G_N$.
- For every j, the point C_j lies on the line L_{ij} if and only if the literal v_i appears in the clause C_j. Further, the point C_j lies on the line $\overline{L}_{ij}$ if and only if the literal $\overline{v_i}$ appears in the clause C_j.

We claim that all the clauses $C_1, \ldots, C_M$ are satisfied if and only if the points $\{C_1, \ldots, C_M\} \cup G_1 \cup \cdots \cup G_N$ can be covered by $N \cdot M$ lines, in which case, for each variable v_i, either all the M lines $L_{i1}, \ldots, L_{iM}$ are present (i.e. v_i is true), or all the M lines $\overline{L}_{i1}, \ldots, \overline{L}_{iM}$ are present (i.e. v_i is false).

We extend the above reduction as follows. For each variable v_i, we add a point at each line L_{ij}, thus forming a set of M new points denoted also by v_i. Further, we add a point at each line $\overline{L}_{ij}$ forming a set of M new points denoted $\overline{v_i}$ (see Figure 8). For each point of $v_i \cup \overline{v_i}$ we draw a line containing that point and so that a grid G'_i of M^2 new points is formed. Using the arguments of Megiddo and Tamir [7], these $2N \cdot M + N \cdot M^2$ new points and $2N \cdot M$ new lines can be added so that the only lines that contain a point of $v_i \cup \overline{v_i}$, for some i, and more than one point in $\{C_1, \ldots, C_M\} \cup G_1 \cup \cdots \cup G_N \cup G'_1 \cup \cdots \cup G'_N$ are precisely the lines in the construction.

Consider now $P = \{C_1, \ldots, C_M\} \cup G_1 \cup \cdots \cup G_N \cup G'_1 \cup \cdots \cup G'_N$, a set of targets, and $F = v_1 \cup \cdots \cup v_N \cup \overline{v_1} \cup \cdots \cup \overline{v_N}$, a set of floodlights. We claim that all the clauses $C_1, \ldots, C_M$ are satisfied if and only the floodlights can be oriented to illuminate P with angle $\alpha = 0$. This is equivalent to claiming that in our construction all the points of P can be covered by $2N \cdot M$ lines. The result thus follows. □

Since deciding $\alpha = 0$ in the *Minimum Angle Illumination Problem* is NP-complete, then the problem does not admit any polynomial-time factor-approximation algorithm. Nevertheless, we can approximate the minimum angle α with the minimum angle α' such that there exists an orientation of the floodlights that cover the whole plane. Assume $m \geq 2$. Using the result of Bose et al. [1] (precisely Theorem 1) we can orient each of the m floodlights of F with angle $\alpha' = 2\pi/m \leq \pi$ such that the whole plane is illuminated and thus P, in $O(m \log m)$ time. This can be ensured because the total sum of the angles is equal to 2π. As a further comment, observe that Theorem 3 implies that the zero dark time decision problem studied by Fusco and Gupta[5] is NP-complete even if all sensors have exactly two orientations, and the sensing region of each of them is a half-line.

5 Conclusions and open problems

Many real-world applications in security, surveillance, and reconnaissance tasks require multiple targets to be monitored using mobile sensors. Given a set of points in the plane, $P \cup F$, where

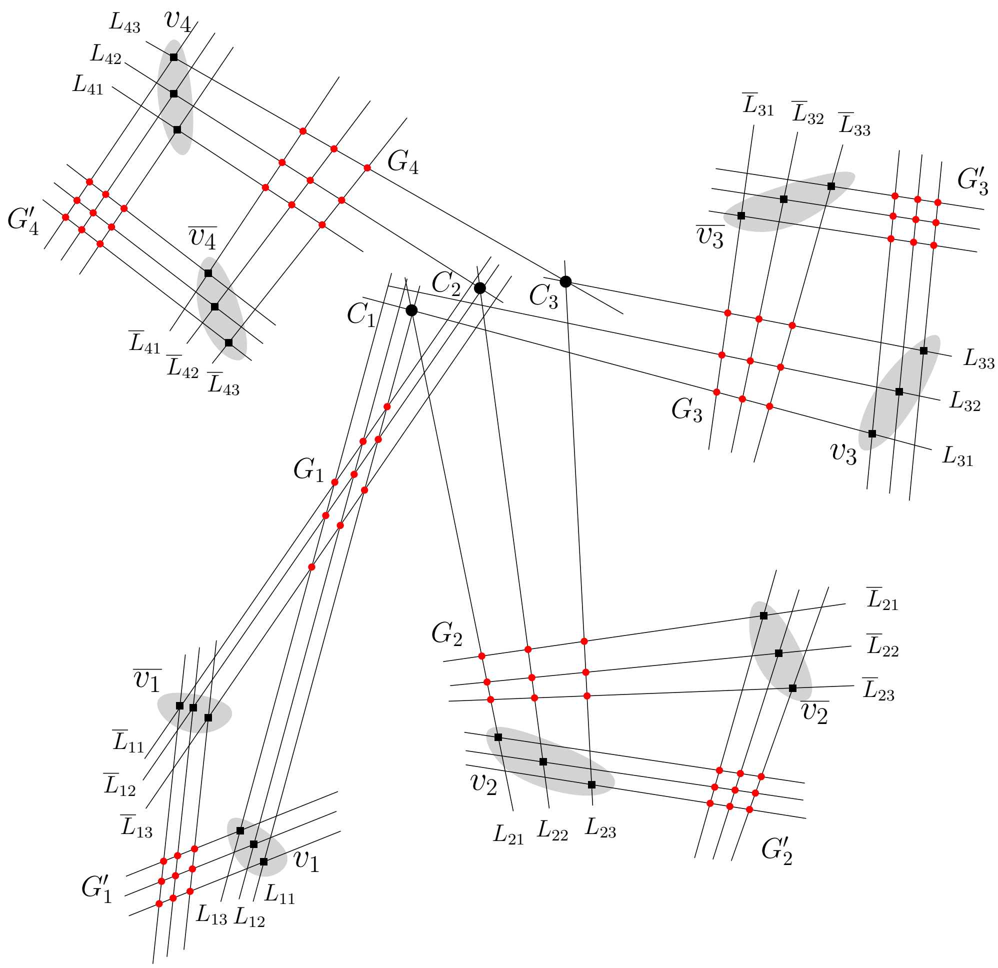


Figure 8: Proof of Theorem 3: The construction for the clauses $C_1 = v_1 \vee v_2 \vee v_3$, $C_2 = \overline{v_1} \vee v_2 \vee v_4$, and $C_3 = v_2 \vee v_3 \vee v_4$, over the variables v_1, v_2, v_3, v_4.

P represents n target points and F represents m floodlights, we have introduced the *Rotating Floodlights Illumination Problem*: compute the minimum angle $\alpha(P, F)$ so that the lights can be scheduled to provide uninterrupted illumination of P as they rotate with identical speeds. We studied the problem of finding tight bounds on the angle $\alpha(P, F)$ in both the 1-dimensional and the 2-dimensional version where there are exactly two floodlights.

Although the 1-dimensional continuous version of the problem can be easily solved[6], the discrete version is trickier. Nonetheless, the optimal angle has been determined in some cases. When the points or floodlights are not necessarily on a line (i.e. the 2-dimensional problem) we presented results for the case in which we have two floodlights to schedule ($m = 2$). However, nothing is known for $n, m \geq 3$, except that $\alpha = \pi$ is always sufficient to cover the whole plane, and thus, it is also sufficient to cover P when $m = 3$ (see [6]). We conjecture that our 2-dimensional Rotating Floodlights Illumination Problem with $n, m \geq 2$ is NP-hard in general, a fact that we hope to establish in the future, in addition to improving the known bounds.

The problems introduced here provide opportunities for further studies, including interesting and unexplored variations of the discrete version of the problem. These include a 1.5-dimensional variant, where the floodlights lie on a given line and the targets are on the plane, special configurations (floodlights and/or targets on a grid), finite illumination range, etc. Some of them have already been partially or totally considered in the continuous version of this problem[6].

Acknowledgements. The problem studied here was introduced and partially solved during the VI Spanish Workshop on Geometric Optimization, June 2012, El Rocío, Huelva, Spain. The authors would like to thank other participants for helpful comments.

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