Guarding Convex Sets

Jurek Czyzowicz[1], Eduardo Rivera-Campo[2], Jorge Urrutia[3] and Joseph Zaks[4]

1) Département d'Informatique, Université du Québec à Hull, Hull, Québec, Canada.

2) Departamento de Matemáticas, Universidad Autónoma Metropolitana-I, México D.F., México.

3) Department of Computer Science, University of Ottawa, Ottawa, Ontario, Canada.

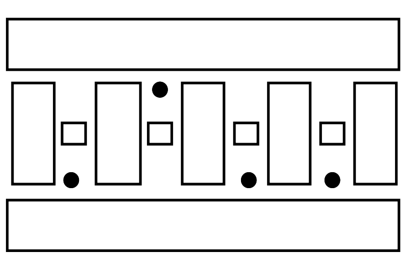
4) Department of Mathematics, University of Haifa, Haifa, Israel.

1. Introduction

Art gallery problems have been intensely studied in the literature in recent years. A typical problem in this area is the following: given a simple polygon P with n vertices in the plane, how many guards are required to completely guard the interior of P? A well known result of Chvatal asserts that $\lfloor n/3 \rfloor$ guards are always sufficient and occasionally necessary. Since then, several versions of this problem have been studied, depending on the shape of the polygon or the type and placements of the guards. A survey of results on the art gallery problems may be found in [5].

In [9], illumination problems for families of disjoint convex sets were studied. In these problems, the typical question is: given a family F of n disjoint compact convex sets, how many light sources, idealized by points on the plane, are needed to completely illuminate the boundaries of the elements of F? It is proved in [9] that 4n-7 points are always sufficient and occasionally necessary to illuminate the boundaries of any family of n disjoint compact convex sets.

In this paper, we study the following problem. Let $F=\{T_1,T_2,\dots,T_n\}$ be a collection of n disjoint compact convex sets on the plane. A set S in R^2 is said to guard F if each of the sets T_i is visible from at least one point in S; i.e. if for every $T_i \in F$ there exists a point x_i in T_i and a point y_i in S such that the segment x_iy_i meets $T_1 \cup T_2 \cup \dots \cup T_n$ only at the point x_i. How many points are needed to guard any collection F of n disjoint compact convex sets in the plane? For example, in Figure 1 we show a collection F of 11 rectangles for which four guards are needed.



Four guards, represented by small circles, are needed to guard these 10 rectangles.

Figure 1

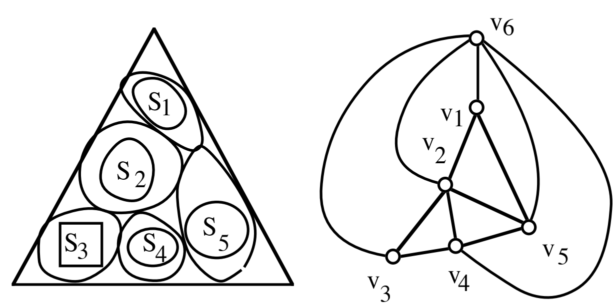


Figure 2

In this paper we prove first that any family $F=\{T_1,T_2,\dots,T_n\}$ of n disjoint plane compact sets can be guarded with at most $\lfloor 2(n-1)/3 \rfloor$ points (guards). This bound is proved to be tight by constructing families of sets for which exactly $\lfloor 2(n-1)/3 \rfloor$ points are required. Some interesting problems arise when the elements of F are restricted to satisfy some extra conditions. We prove that any family of n disjoint of line segments can be guarded using at most $n/2$ points and that occasionally $2n/5$ points are needed. The 3-dimensional case turns out to be different altogether, for in this case it is shown that no constant c, $c<1$, exists such that any family F of n disjoint compact sets can be guarded with cn points.

2. Guarding Plane Convex Sets

We now proceed to give a tight bound on the number of points needed to guard families of arbitrary disjoint compact convex sets on the plane. We prove:

Theorem 1. $\lfloor 2(n-1)/3 \rfloor$ points are always sufficient and occasionally necessary to guard any family $F=\{S_1,S_2,\dots,S_n\}$ of n disjoint convex compact sets in the plane.

The following result will be used in the proof of this theorem:

Theorem N (Nishizeki [6])**:** If G is a planar graph of n nodes with minimum vertex degree $\delta \geq 3$ and with connectivity $k \geq 2$, then for all $n \geq 14$, the number of edges in a maximum matching of G is greater than or equal to $\lceil (n+4)/3 \rceil$ and for $n<14$, the number of edges is $\lfloor n/2 \rfloor$.

Proof of Theorem 1. Let $F=\{S_1,S_2,\dots,S_n\}$ be any family of n disjoint compact convex sets in the plane; we may assume that they are all contained in a *large enough triangular region* T. Let S'_{n+1} be the complement of T.

Construct a family $F'=\{S'_1, S'_2,\dots,S'_n\}$ with n strictly convex sets with pairwise *disjoint interiors* such that :

i) $T \supseteq S'_i \supseteq S_i$, $i=1,2,\dots,n$.

ii) The number of pairs of sets S'_i, S'_j, whose boundaries meet is as large as possible;

It is easy to verify that if T is chosen to be sufficiently large, each S'_i will intersect at least three different sets S'_j, $j\neq i$, $i=1,\dots,n+1$. (See Figure 2).

Define the dual graph D of $S'_1,S'_2,\dots,S'_{n+1}$ as follows: D has a vertex v_i for each set S'_i; the vertices v_i and v_j are adjacent in D if S'_i and S'_j have a point in common. The graph D is planar, 2-connected and with minimum degree at least three. By Theorem N, D has a matching M of size at least $\lceil((n+1)+4)/3\rceil$. To guard $S_1,S_2,\dots,S_n$ it is sufficient to select a point in the common boundary of each pair S'_i, S_j matched in M, plus a point for each of the remaining unmatched sets. This yields a total of $[(n+1)-(2\lceil(n+5)/3\rceil)] + \lceil(n+5)/3\rceil = \lfloor 2(n-1)/3\rfloor$ points which collectively guard F.

The following example shows that as many as $\lfloor 2(n-1)/3\rfloor$ points might be necessary to guard n disjoint discs in the plane.

Start with three mutually tangent discs C_1, C_2 and C_3 and consider the gap formed by them (i.e. the region bounded by them). Insert one disc in this gap tangent to C_1, C_2 and C_3 so as to form three smaller gaps. In each gap insert a smaller disc tangent to the three circles bounding each gap, so as to create nine new gaps.

Continue inserting discs until 3^k gaps are obtained. In the final step, shrink all the discs by an amount $\varepsilon>0$; insert 3^k discs, one in each gap, and 3 more discs outside the gap formed by C_1, C_2 and C_3. This may be done in such a way that no two of these $3^k + 3$ discs are visible from a point. It follows that a guarding set must contain at least $3^k + 3$ points, one for each disc inserted in the last step.

However, the total number of discs is $n = 3 + 1 + 3 + 3^2 + 3^3 + \dots + 3^{k-1} + (3^k + 3) = (3^{k+1} + 11)/2$ and $3^k + 3 = \lfloor 2(n-1)/3\rfloor$.

Corollary 1. $\lfloor 2(n-1)/3\rfloor$ points are always sufficient and occasionally necessary to guard any family $F=\{S_1,S_2,\dots,S_n\}$ of n disks.

3. Line Segments

In this section we turn our attention to the study of guarding problems for families of line segments. Our main objective in this section is to prove the following result:

Theorem 2. Any collection of n line segments can always be guarded using at most $\lfloor n/2\rfloor$ points; $\lfloor(2n-9)/5\rfloor$ points are occasionally necessary.

Some preliminary results and definitions will be needed before we can prove our result.

Consider a collection $F=\{S_1,\dots, S_n\}$ of n disjoint compact convex sets on the plane. Construct a graph G(F) with n vertices $v_1,\dots v_n$ such that v_i is adjacent to v_j iff there is a point x on the plane that sees at least a point in the boundary of each one of S_i and S_j. In Figure 3 a collection F of eight line segments and its corresponding graph G(F) is shown.

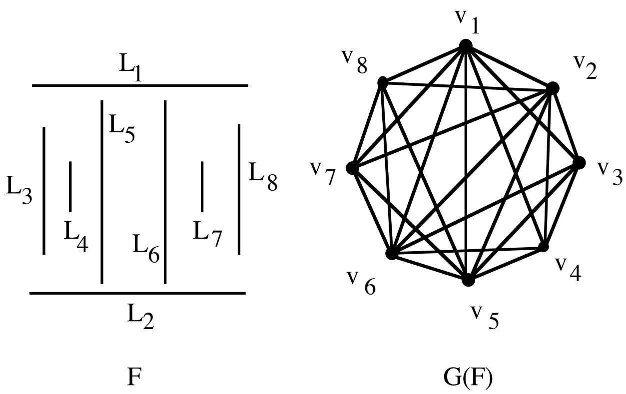


Figure 3

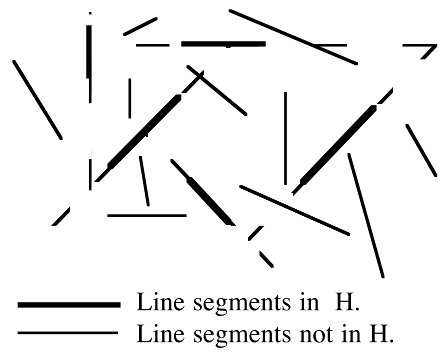


Figure 4

The main idea in our proof of Theorem 2 is to show that for any family F with an even number n of disjoint line segments, $G(F)$ has a perfect matching. We recall a well-known result of Tutte which provides necessary and sufficient conditions for the existence of perfect matchings in graphs.

Theorem T (Tutte). A graph G has a perfect matching iff for every subset S of V(G), $Odd(G\text{-}S) \leq |S|$.

The following lemma, given without proof, will be used to prove our main result:

Lemma 1. Let Q be any convex polygon, $F=\{S_1,..., S_n\}$ a family of n disjoint line segments and H the subset of elements of F containing at least one point in Q. Then the subgraph of G(F) induced by the vertices of G(F) representing elements in H is connected.

We proceed now to prove Theorem 2.

Proof of Theorem 2: Let $F=\{L_1,..., L_n\}$ be a collection of n disjoint line segments and G(F) its associated graph. Assume that n is even, otherwise add any other line segment to F. We now show that G(F) satisfies Tutte's theorem and thus has a perfect matching M. Consider any subset H of F and let us call S the set of vertices of G(F) representing elements of H. We now show that the number of connected components of G(F)-S is at most $|S|=|H|$.

To start, delete from the plane all the line segments *not* in H. One at a time, extend the elements of H until they meet another element of H, meet a previously extended element of H or become lines or semilines. Let π be the plane partition induced by the extended elements of H. It is easy to verify that π contains exactly $|H|+1$ polygonal faces. Replace the elements of F not in H. (See Figure 4).

By Lemma 1, the number of components of G(F)-S is at most the number of faces of π, which is $|H|+1$. The reader can easily verify that there are at least two adjacent faces in π such that the line segments that intersect them are in the same component in G(F)-S. Then the number of components in G(F)-S is at most |S| and G(F) has a perfect matching M. It now follows that F can be illuminated using at most n/2 points on the plane, one for each edge in M.

We now construct an example of a family F with n line segments such that $\lfloor (2n-9)/5 \rfloor$ points are required to guard F. Let H be a cubic plane graph with a triangular outer face T. H may be redrawn on the plane using straight line segments to represent its edges in such a way that for all of its vertices v, except the outer ones, the three vectors emanating from the vertex v along the edges positively span the plane. Let H have k vertices; it has 3k/2 edges. Substitute the edges of H by line segments such that at each of the k-3 inner vertices we obtain a triangular face in which we insert a small segments.

Discard the 3 edges of the outer face of H and disconnect each edge in a small neighborhood of its end vertices to form a collection of n=(3k/2) + k - 3 segments. Since no two of our k-3 small segments are visible from a single point, k-3 points are needed to guard our collection of line segments. It is easy to verify that k - 3 points are also sufficient. But k-3 = $\lfloor (2n-9)/5 \rfloor$ thus proving our result.

When all of our line segments are parallel to the x or y-axes, we have been unable to improve on the general upper bound of n/2 for guarding line segments.

3. Guarding Convex Sets in R^3

We finish by remarking that the situation in R^3 is quite different. In fact, there exist no constants c and k, with c < 1, for which it is true that every collection of n mutually disjoint compact convex sets can be guarded from cn+k points. To see this, we repeat a construction given in [2], (see also [5]).

Given three intervals I, J, K of the real line, let Box(I,J,K) be the box of all points $(x,y,z) \in R^3$ such that $x \in I$, $y \in J$, $z \in K$. Consider the families $X=\{[2i+\varepsilon, 2i+1-\varepsilon]: i=0,\dots,m\}$, $Y=\{[2i-1+\varepsilon, 2i-\varepsilon]: i=1,\dots,m\}$ and $Y'=\{[2i-1-\varepsilon, 2i+\varepsilon]: i=1,\dots,m\}$.

Let $A=\{Box(I_a,J_b,K_c) : I_a,J_b \in X, K_c=[0,2m+1]\}$, $B=\{Box(I_a,J_b,K_c) : I_a=[0,2m+1], J_b \in Y, K_c \in X\}$, $C=\{Box(I_a,J_b,K_c) : I_a, K_c \in Y, J_b=[0,2m+1]\}$ and $F'=A \cup B \cup C$. Notice that the elements of F' leave a set of $O(m^3)$ gaps (small cubes) between them. In half of them, namely the ones of the form $Box(I_a,J_b,K_c) : I_a, J_b, K_c \in Y'$, insert a small box in the center of them.

There are m^3 tiny boxes, and $3m^2 + 3m+1$ long boxes. However, no two of the m^3 tiny boxes are visible from a common point. Therefore at least m^3 points are needed to guard these $m^3 + 3m^2 + 3m+1$ boxes. Thus we have proved:

Theorem 3. There is no constant c<1 such that any family with n boxes can be guarded with cn points.

4. Conclusions

In this paper we proved that $\lfloor 2(n-1)/3 \rfloor$ points are always sufficient and occasionally necessary to guard any family F of n disjoint compact convex sets. For line segments, we proved that $\lfloor n/2 \rfloor$ points are always sufficient and that $\lfloor (2n-9)/5 \rfloor$ points are occasionally sufficient. It is not hard to construct families F of line segments containing only line segments line segments parallel to the x or y-axes for which n/3 points are needed to guard F. We would like to state the following conjectures:

Conjecture 1. $\lfloor (2n-9)/5 \rfloor$ points are always sufficient to guard n disjoint line segments.

Conjecture 2. $n/3 \pm c$ points are always sufficient to guard any family of n disjoint orthogonal line segments.

References

[1] C. Berge, Sur le couplage maximum d'un graphe, *C.R. Acad. Sciences*, Paris, **247**, 258-259, 1958.

[2] J. Czyzowicz, E. Rivera-Campo, N. Santoro, J. Urrutia and J. Zaks, Guarding rectangular art galleries, Technical Report TR-89-27, Univ. of Ottawa, August 1989.

[3] J. Czyzowicz, E. Rivera-Campo and J. Urrutia, Illuminating triangles and rectangles on the plane, To appear in *Journal of Combinatorial Theory B*.

[4] J. Czyzowicz, E. Rivera-Campo, J. Urrutia and J. Zaks, Illuminating lines and circles on the plane, Technical Report TR-89-49, Univ. of Ottawa, December 1989.

[5] J. O'Rourke, *Art Gallery Theorems and Algorithms*, Oxford Press, 1987.

[6] T. Nishizeki, Lower bounds on cardinality of the maximum matchings of planar graphs, Carnegie Mellon Tech. Report, 1977.

[7] W. Tutte, A theorem on planar graphs, *Trans. Amer. Math. Soc.* **82**, 99-116, 1956.

[8] W. Tutte, The factorization of linear graphs, *J. London Math. Soc.* **22**, 107-111, 1947.

{9} J. Urrutia, and J. Urruria, Iluminating convex sets, Technical Reprt TR-89-31, University of Ottawa, 1989.