Equal Area Polygons in Convex Bodies

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Abstract. In this paper, we consider the problem of packing two or more equal area polygons with disjoint interiors into a convex body K in $\boldsymbol{E}^2$ such that each of them has at most a given number of sides. We show that for a convex quadrilateral K of area 1, there exist n internally disjoint triangles of equal area such that the sum of their areas is at least $\frac{4n}{4n+1}$. We also prove results for other types of convex polygons K. Furthermore we show that in any centrally symmetric convex body K of area 1, we can place two internally disjoint n-gons of equal area such that the sum of their areas is at least $\frac{n-1}{\pi}\sin\frac{\pi}{n-1}$. We conjecture that this result is true for any convex bodies.

1 Introduction

For a subset S of $\boldsymbol{E}^2$ having a finite area, let $A(S)$ denote the area of S. A compact convex set with nonempty interior is called a *convex body.*

In [2], W. Blaschke showed the following theorem:

Theorem A. *Let K be a convex body in $\boldsymbol{E}^2$, and let T be a triangle with maximum area among all triangles contained in K. Then $\frac{A(T)}{A(K)} \geq \frac{3\sqrt{3}}{4\pi}$ with equality if and only if K is an ellipse.*

E. Sás [13] generalized Blaschke's result as follows:

Theorem B. *Let K be a convex body in $\boldsymbol{E}^2$, and let P be a polygon with maximum area among all polygons contained in K and having at most n sides. Then $\frac{A(P)}{A(K)} \geq \frac{n}{2\pi}\sin\frac{2\pi}{n}$ with equality if and only if K is an ellipse.*

For subsets $A_1, \cdots, A_m$ of $\boldsymbol{E}^2$, we say that the A_i are *internally disjoint* if the interiors of any two A_i and A_j with $1 \leq i < j \leq m$ are mutually disjoint. In this paper, we consider the problem of packing two or more equal area internally

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disjoint polygons in a convex body in $\boldsymbol{E}^2$ such that each of them has at most a given number of sides, and the sum of their areas is maximized.

Let K be a convex body in $\boldsymbol{E}^2$ and let $\mathcal{P}_{m,n}(K)$ denote a family of m internally disjoint equal area convex polygons $P_1, \cdots, P_m \subset K$ such that each P_i, $1 \le i \le m$, has at most n sides, and define

$$s(K; m, n) = \sup_{\{P_1, \cdots, P_m\} \in \mathcal{P}_{m,n}(K)} \frac{A(P_1) + \cdots + A(P_m)}{A(K)}.$$

We simply write $t_m(K)$ for $s(K; m, 3)$. Clearly $t_m(T) = 1$ for any triangle T and positive integer m, and hence $s(T; m, n) = 1$ for any triangle T and integers $m \ge 1$ and $n \ge 3$. In general, for any integers k, m, n with $n \ge k \ge 3$, $m \ge 1$ and for any convex polygon K with at most k sides, $s(K; m, n) = 1$ (Fig. 1).

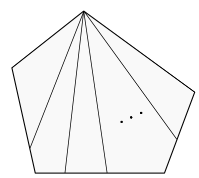

Fig. 1.

Monsky [10] showed that a rectangle can be dissected into m equal area triangles if and only if m is even. Thus

Theorem C. *Let m be a positive integer and let R be a rectangle. Then $t_m(R) = 1$ for any even integer m and $t_m(R) < 1$ for any odd integer m.*

Furthermore, Kasimatis showed that a regular k-gon, $k \ge 5$, can be dissected into m equal area triangles if and only if m is a multiple of k [6]; and Kasimatis and Stein showed that almost all polygons cannot be dissected into equal area triangles [7]:

Theorem D. *Let k be an integer with $k \ge 5$ and let K be a regular k-gon. Then $t_m(K) = 1$ for any positive integer $m \equiv 0 \pmod k$ and $t_m(K) < 1$ for any positive integer $m \not\equiv 0 \pmod k$.*

Theorem E. *For almost all polygons K and for any integer $m \ge 1$, $t_m(K) < 1$.*

2 Preliminary Results

We now show some propositions that will be needed to prove our results . For a subset S of $\boldsymbol{E}^n$, we denote the convex hull of S by $conv(S)$.

Proposition 1. *Let n be an integer with $n \geq 3$, P a convex polygon with at least n sides, and let α denote the value of the maximum area of a convex polygon contained in P with at most n sides. Then there exists an n-gon of area α each of whose vertices is a vertex of P.*

Proof. Let $P = p_1p_2\cdots p_k$, $k \geq n$. Take a convex polygon $Q \subseteq P$ with at most n sides such that $A(Q) = \alpha$ and the number of common vertices of P and Q is maximized. By way of contradiction, suppose that there is a vertex a of Q such that $a \notin \{p_1, \cdots, p_k\}$. By the maximality of $A(Q)$, a is on the boundary of P, and hence a is an interior point of a side of P. We may assume $a \in p_1p_2 - \{p_1, p_2\}$. Let b and c be distinct vertices of Q adjacent to a. Then $A(abc) \leq \max\{A(p_1bc), A(p_2bc)\}$. We may assume $A(abc) \leq A(p_1bc)$. Let $Q' = \mathrm{conv}((Q - abc) \cup p_1bc)$. Then $Q' \subseteq P$, Q' has at most n sides, $\alpha = A(Q) \leq A(Q')$ (so $\alpha = A(Q')$ by the maximality of α), and the number of common vertices of Q' and P is strictly greater than that of Q and P, a contradiction. Thus any vertex of Q is a vertex of P, and it follows from the maximality of α that Q has n sides. □

Proposition 2. *Let K be a convex body in $\boldsymbol{E}^2$ and let m and n be integers with $m \geq 3$ and $n \geq 3$. Suppose that K contains internally disjoint polygons $P = p_1\cdots p_m$ and $Q = q_1\cdots q_n$. Then K contains internally disjoint polygons P' and Q' such that $\mathrm{conv}(P' \cup Q')$ has at most $m+n-2$ sides, P' has at most m sides, Q' has at most n sides, and $A(P') \geq A(P)$, and $A(Q') \geq A(Q)$.*

Remark 1. A simple proof for the case where $m = n = 3$ is shown in [12].

Proof. Let $S = \mathrm{conv}(P \cup Q)$. If S has at most $m+n-2$ sides, then we have only to let $P' = P$ and $Q' = Q$. Thus assume that S has $m+n$ sides or $m+n-1$ sides.

Case 1. S has $m+n$ sides:

We may assume that $S = p_1p_2\cdots p_mq_1q_2\cdots q_n$ and that the straight line l passing through p_1 and parallel to p_2q_{n-1} satisfies the condition that $(l \cap p_1p_2q_{n-1}q_n) - \{p_1\} \neq \emptyset$ (Fig. 2 (a)). Let r be the intersection point of p_1p_m and p_2q_{n-1}. Then $A(q_np_2r) \geq A(p_1p_2r)$, and hence $P^* = q_np_2p_3\cdots p_m$ is a convex polygon with m sides such that P^* is internally disjoint to Q and $A(P^*) \geq A(P)$. Using the same arguments for Q^* and Q, we obtain P' and Q' with the desired properties.

Case 2. S has $m+n-1$ sides:

We may assume that $S = p_1p_2\cdots p_{m-1}q_1q_2\cdots q_n$ and that $A(p_1p_{m-1}q_1) \geq A(p_1p_{m-1}q_n)$ (Fig. 2 (b)). Then $A(p_1p_{m-1}q_1) \geq A(p_1p_{m-1}p_m)$, and hence $P^* = p_1p_2\cdots p_{m-1}q_1$ is a convex polygon with m sides such that P^* is internally disjoint to Q and $A(P^*) \geq A(P)$. Proceeding the same way for Q^* we obtain P' and Q' with the desired properties. □

Proposition 3. *Let $P = p_1p_2p_3p_4p_5$ be a convex pentagon with $A(P) = 1$ and let $\alpha = \frac{5-\sqrt{5}}{10}$. Then there exist indices i and j such that $A(p_{i-1}p_ip_{i+1}) \leq \alpha \leq A(p_{j-1}p_jp_{j+1})$ (indices are taken modulo 5).*

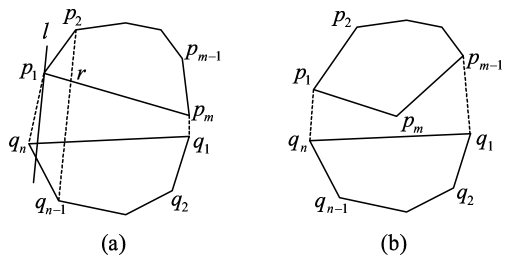

Fig. 2.

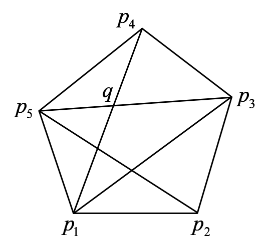

Fig. 3.

Proof. We first show that there exists an index i such that $A(p_{i-1}p_ip_{i+1}) \leq \alpha$. By way of contradiction, suppose that $A(p_{i-1}p_ip_{i+1}) > \alpha$ for any i with $1 \leq i \leq 5$. Then $A(p_1p_2p_3) > \alpha$, $A(p_1p_2p_5) > \alpha$ and $A(p_1p_2p_4) < 1 - 2\alpha$. Let q be the intersection point of p_1p_4 and p_3p_5 (Fig. 3). Since $A(p_1p_2q) \geq \min\{A(p_1p_2p_3),\ A(p_1p_2p_5)\} > \alpha$, $\frac{p_1q}{p_1p_4} = \frac{A(p_1p_2q)}{A(p_1p_2p_4)} > \frac{\alpha}{1-2\alpha}$. Therefore $A(p_3p_4p_5) = (1 - A(p_1p_2p_3)) \times \frac{qp_4}{p_1p_4} < (1-\alpha) \times \frac{1-3\alpha}{1-2\alpha}$. On the other hand, we have $A(p_3p_4p_5) > \alpha$ by assumption. Consequently, $\alpha < \frac{(1-\alpha)(1-3\alpha)}{1-2\alpha}$, and hence we must have $5\alpha^2 - 5\alpha + 1 > 0$. This contradicts $\alpha = \frac{5-\sqrt{5}}{10}$. Similarly we can verify that there exists an index j such that $A(p_{j-1}p_jp_{j+1}) \geq \alpha$. □

We conclude this section with two more propositions shown in [12]. Proposition 4 is obtained by using the *Ham Sandwich Theorem* (see, for example, [9, 14]) and a small adjustment, and Proposition 5 is obtained by using an extension of the Ham Sandwich Theorem shown in [1, 5, 11]:

Proposition 4. *Let n be an integer with $n \geq 3$ and let K be a convex polygon with at most n sides. Then $s(K, 2, \lfloor \frac{n}{2} \rfloor + 2) = 1$.*

Proposition 5. *Let n be an integer with $n \geq 3$ and let K be a convex polygon with at most n sides. Then $s(K, 3, \lceil \frac{n}{3} \rceil + 4) = 1$.*

Remark 2. Combining Propositions 4 and 5, we obtain several results. For example, for a convex polygon K with at most $k = 2^l+3$ sides, we have $s(K; 1, 2^l+3) = s(K; 2, 2^{l-1} + 3) = s(K; 2^2, 2^{l-2} + 3) = \cdots = s(K; 2^l, 4) = s(K; 2^{l+1}, 4) = s(K; 2^{l+2}, 4) = \cdots = 1$ (and $s(K; 2^{l+1}, 3) \geq \frac{8}{9}$ by the equality $s(K; 2^l, 4) = 1$ and Theorem 2 to be shown in Section 3); for a polygon K with at most $k = 3^l+r$ sides, $r \in \{6, 7\}$, we have $s(K; 1, 3^l+r) = s(K; 3, 3^{l-1}+r) = s(K; 3^2, 3^{l-2}+r) = \cdots = s(K; 3^l, 1+r) = s(K; 3^{l+1}, 7) = s(K; 3^{l+2}, 7) = \cdots = 1$; for a polygon with at most 30 sides, $s(K; 3, 14) = s(K; 6, 9) = s(K; 12, 6) = 1$; and so on.

3 Equal Area Polygons in a Convex Polygon

Theorem 1. *Let K be a convex body in $\boldsymbol{E}^2$ and let $\boldsymbol{u}$ be a non-zero vector in $\boldsymbol{E}^2$. Then there exist internally disjoint equal area triangles T_1 and T_2 in K such that $T_1 \cap T_2$ is a segment parallel to $\boldsymbol{u}$ and $A(T_1) + A(T_2) \geq \frac{1}{2}A(K)$.*

Proof. Let l_1 and l_2 be distinct straight lines, each of which is parallel to $\boldsymbol{u}$ and tangent to K (Fig. 4). Let a be a contact point of l_1 and K and let b be a contact point of l_2 and K. Let m be the midpoint of the segment ab, and let c and d be intersection points of the perimeter of K and the straight line passing through m and parallel to $\boldsymbol{u}$. Let e and g be the intersection points of the straight line tangent to K at c and straight lines l_1 and l_2, respectively, and let f and h be the intersection points of the straight line tangent to K at d and straight lines l_1 and l_2, respectively. Let l_3, l_4 be straight lines perpendicular to $\boldsymbol{u}$ and passing through c, d, respectively, and label the vertices of the rectangle surrounded by l_1, l_2, l_3 and l_4, as shown in Fig. 4. Then for triangles $T_1 = acd$ and $T_2 = bcd$, $T_1 \cap T_2$ is a segment parallel to $\boldsymbol{u}$, $A(T_1) = A(T_2)$, and it follows from the convexity of K that

$$A(T_1) + A(T_2) = \frac{1}{2}A(e'g'h'f') = \frac{1}{2}A(eghf) \geq \frac{1}{2}A(K),$$

as desired. $\square$

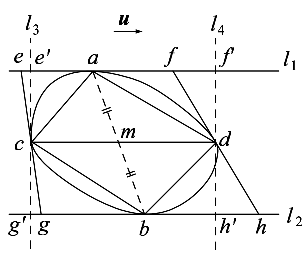

Fig. 4.

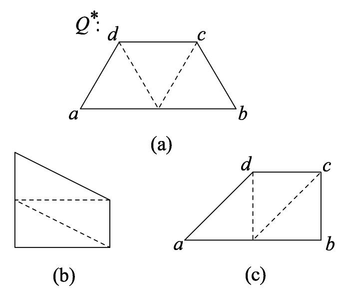

Fig. 5.

Theorem 2. *Let K be a convex quadrilateral. Then the following hold:*

(i) $t_2(K) \geq \frac{8}{9}$ with equality if and only if K is affinely congruent to the quadrilateral Q^* shown in Fig. 5 (a); and
(ii) $t_n(K) \geq \frac{4n}{4n+1}$ for any integer $n \geq 2$.

Proof. (i) Let $K = p_1p_2p_3p_4$. We may assume

$$A(p_1p_2p_4) \geq A(p_1p_2p_3) \text{ and } A(p_1p_2p_4) \geq A(p_1p_3p_4). \tag{1}$$

By considering a suitable affine transformation f, we may assume further that $f(p_1) = O(0,0)$, $f(p_2) = a(1,0)$, $f(p_4) = c(0,1)$ (Fig. 6 (a)). Write $f(p_3) = b$, let $e = (1,1)$ and let m be the midpoint of ac. By (1) and the convexity of K, $b \in ace$. By symmetry, we may assume that $b \in ame$. Let d be the intersection point of the straight lines Om and bc. Then d is on the side bc and $A(Oad) = A(Ocd)$. We show that $A(Oad) + A(Ocd) \geq \frac{8}{9}A(K)$. For this purpose, we let b' be the intersection point of the straight lines bc and $x = 1$ (Fig. 6 (b)), and we show that $2A(Oad) \geq \frac{8}{9}A(Oab'c)$.

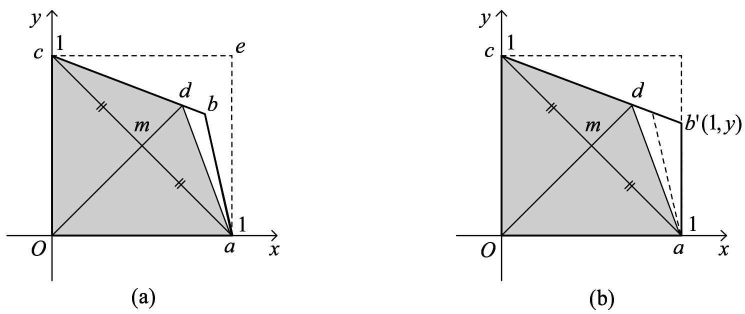

Fig. 6.

Write $b' = (1, y)$. We have $0 < y \leq 1$, $A(Oab'c) = \frac{y+1}{2}$. Furthermore, since $d = \left(\frac{1}{2-y}, \frac{1}{2-y}\right)$, $2A(Oad) = \frac{1}{2-y}$. Hence,

$$\frac{2A(Oad)}{A(Oab'c)} = \frac{2}{(2-y)(y+1)} = \frac{2}{-\left(y-\frac{1}{2}\right)^2 + \frac{9}{4}} \geq \frac{8}{9},$$

as desired.

Next we show that for a convex quadrilateral K, $t_2(K) = \frac{8}{9}$ holds if and only if K is affinely congruent to Q^*. If $t_2(K) = \frac{8}{9}$, then, in the argument above, we must have $b = b'$ and $y = \frac{1}{2}$. Hence $t_2(K) = \frac{8}{9}$ implies that K is affinely congruent to the quadrilateral shown in Fig. 5 (b), and hence to Q^*. Now we show that for a convex quadrilateral K affinely congruent to Q^* and for any choice of two internally disjoint equal area triangles T_1 and T_2 in K, $\frac{A(T_1)+A(T_2)}{A(K)} \leq \frac{8}{9}$. It suffices to show this for the case where $K = Q^*$, whose vertices are labeled as shown in Fig. 5 (a). Let T_1 and T_2 be internally disjoint equal area triangles in K, and let l be a straight line such that each of the half-planes H_1 and H_2 with $H_1 \cap H_2 = l$ contains one of T_1 or T_2. Let p and q be the intersection points of l and the perimeter of K. Four cases arise:

(a) $\{p, q\} \in ab \cup bc$ or $\{p, q\} \in ab \cup da$;
(b) $\{p, q\} \in ab \cup cd$;
(c) $\{p, q\} \in bc \cup cd$ or $\{p, q\} \in cd \cup da$;
(d) $\{p, q\} \in bc \cup da$.

First consider case (b). We may assume $p \in ab, q \in cd$ and $T_1 \subseteq apqd$. Write $S = A(apq) = A(apd)$ and $S' = A(aqd) = A(pqd)$. By Proposition 1, $A(T_1) \le S$ or $A(T_1) \le S'$. If $A(T_1) \le S$, then we can retake T_1 in apd, and this case is reduced to Case (a). If $A(T_1) \le S'$, then we can retake T_1 in aqd, and this case is reduced to Case (c). Next consider Case (c). By symmetry, we consider only the case when $\{p, q\} \in bc \cup cd$. We may assume $p \in bc, q \in cd$ and $T_1 \subseteq cpq$. Then $A(T_1) \le A(bcd) \le \frac{1}{3}A(K)$. Hence $A(T_1) + A(T_2) \le \frac{2}{3}A(K) < \frac{8}{9}A(K)$ in this case. Next consider Case (d). We may assume $p \in bc, q \in da$ and $T_1 \subseteq abpq$. By symmetry, we may assume further that $bp \ge aq$. Then since $A(T_1) \le A(abp)$, we can retake T_1 in abp, and hence this case is reduced to Case (a).

Finally, we consider Case (a). By symmetry, we consider only the case where $\{p, q\} \in ab \cup bc$. Furthermore, by considering a suitable affine transformation, we may assume that $K = abcd$ is the trapezoid shown in Fig. 5 (c) with $bc = 1$, $p \in ab, q \in bc$ and $T_1 \subseteq bpq$. We show that $A(T_1) + A(T_2) \le \frac{8}{9}A(K) = \frac{4}{3}$. For this purpose, we suppose that $A(T_1) \ge \frac{2}{3}$ and show that $A(T_2) \le \frac{2}{3}$. In view of Proposition 1, it suffices to show that any triangle whose vertices are in $\{a, p, q, c, d\}$ has area at most $\frac{2}{3}$. Let $x = bp$ and let $y = bq$. Since $\frac{2}{3} \le A(T_1) \le A(bpq)$ by assumption, $x \ge \frac{4}{3}$ and $y \ge \frac{2}{3}$. Hence $A(cdq) \le A(cdp) \le A(qdp) \le A(qda) = A(abcd) - (A(abq) + A(cdq)) = 1 - \frac{y}{2} \le \frac{2}{3}$, $A(acd) = \frac{1}{2}$, $A(apq) \le A(apc) = A(apd) = \frac{2-x}{2} \le \frac{1}{3}$ and $A(cpq) \le A(caq) = 1 - y \le \frac{1}{3}$. Thus we have $A(T_2) \le \frac{2}{3}$, as desired.

(ii) Let $K = p_1p_2p_3p_4$. We may assume that $A(K) = 1$ and $A(p_1p_2p_3) \ge \frac{1}{2}$. We show (ii) by induction on n. Suppose that $t_n(K) \ge \frac{4n}{4n+1}$ for some $n \ge 2$. Take point q on p_2p_3 such that $A(p_1p_2q) = \frac{4}{4(n+1)+1} \left(< \frac{1}{2}\right)$. By induction, there exist n internally disjoint triangles $T_1, \cdots, T_n$ in $p_1qp_3p_4$ such that $A(T_1) = \cdots = A(T_n) = \frac{4}{4n+1} \times A(p_1qp_3p_4) = \frac{4}{4(n+1)+1} = A(p_1p_2q)$. Thus $t_{n+1}(K) \ge \frac{4(n+1)}{4(n+1)+1}$, as desired. □

Theorem 3. *Let K be a convex pentagon. Then the following hold:*

(i) $t_2(K) \ge \frac{2}{3}$*;*
(ii) $t_3(K) \ge \frac{3}{4}$*; and*
(iii) $t_n(K) \ge \frac{2n}{2n+1}$ *for any integer* $n \ge 4$*.*

Proof. Let $K = p_1p_2p_3p_4p_5$. We may assume that

$$A(p_1p_2p_5) \ge A(p_ip_{i+1}p_{i+2}) \quad \text{for} \quad 1 \le i \le 4, \tag{2}$$

where $p_6 = p_1$.
(i) By considering a suitable affine transformation f, we may assume that $f(p_1) = O(0, 0)$, $f(p_2) = a(1, 0)$, $f(p_5) = d(0, 1)$. Write $f(p_3) = b(x_1, y_1)$, $f(p_4) = c(x_2, y_2)$ (Fig. 7). We have

$$2A(Oad) = 1, \quad 2A(Oab) = y_1, \quad 2A(Ocd) = x_2, \tag{3}$$

$$\left.\begin{aligned} 2A(abc) &= |\overrightarrow{ab} \times \overrightarrow{ac}| = (x_1y_2 - y_1x_2) + (y_1 - y_2) \text{ and} \\ 2A(bcd) &= |\overrightarrow{db} \times \overrightarrow{dc}| = (x_1y_2 - y_1x_2) + (x_2 - x_1). \end{aligned}\right\} \tag{4}$$

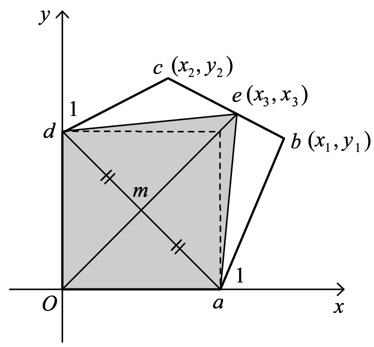

Fig. 7.

Since $A(Oab) \le A(Oad)$ and $A(Ocd) \le A(Oad)$ by (2), it follows from (3) that

$$0 < y_1 \le 1 \quad \text{and} \quad 0 < x_2 \le 1. \tag{5}$$

Furthermore, if there exists a triangle $T \in \{Oab, abc, bcd, Ocd\}$ having area at most $\frac{1}{4}A(K)$, then applying Theorem 2 to the quadrilateral $K - T$, we obtain $t_2(K) \ge \frac{8}{9} \cdot \frac{3}{4} = \frac{2}{3}$, as desired. Therefore we may, in particular, assume that

$$A(Oab) + A(Ocd) > \tfrac{1}{2}A(K) \text{ and} \tag{6}$$
$$A(abc) + A(bcd) > \tfrac{1}{2}A(K). \tag{7}$$

Since (6) implies $A(Obc) < \frac{1}{2}A(K)$, it follows from (7) that $2[A(abc)+A(bcd)] > 2A(Obc) = |\overrightarrow{Ob} \times \overrightarrow{Oc}| = x_1y_2 - y_1x_2$, and hence

$$(x_1y_2 - y_1x_2) + (x_2 - x_1) + (y_1 - y_2) > 0 \tag{8}$$

by (4). Let m be the midpoint of ad and let $e(x_3, x_3)$ be the intersection point of the straight lines Om and bc. Then $x_3 = \frac{x_1y_2 - y_1x_2}{x_1 - x_2 + y_2 - y_1}$, and hence

$$x_3 > 1 \tag{9}$$

by (8). Thus e is on the side bc, and Oae and Ode are equal area triangles in K. We show $A(Oae) + A(Ode) > \frac{2}{3}A(K)$. Write $A(Oad) = S_1$, $A(ead) = S_2$. Then $S_2 = \frac{em}{Om}S_1 > S_1$ by (9). Furthermore, since $A(abe) + A(cde) \le \max\{A(abc), A(bcd)\}$, it follows from (2) that $(S_2 >)S_1 \ge A(abe) + A(cde)$. Consequently, $\frac{A(abe)+A(cde)}{A(K)} < \frac{1}{3}$, and hence $\frac{A(Oae)+A(Ode)}{A(K)} > \frac{2}{3}$, as desired.

(ii) Let $\mathcal{P}$ be the set of convex pentagons, and let $\tau = \inf_{P \in \mathcal{P}} t_2(P)$. We first show that $\frac{\tau}{\tau+2} < \frac{5-\sqrt{5}}{10}$. Let $P = r_1r_2r_3r_4r_5$ be a regular pentagon. In view of Propositions 2 and 1 it follows that $\tau \le t_2(P) \le \frac{A(r_1r_2r_3r_4)}{A(P)} = \frac{5+\sqrt{5}}{10}$. Thus $\frac{\tau}{\tau+2} = 1 - \frac{2}{\tau+2} \le \frac{5+\sqrt{5}}{25+\sqrt{5}} < \frac{5-\sqrt{5}}{10}$.

Now consider any convex pentagon $K = p_1p_2p_3p_4p_5$ of area 1. In view of Proposition 3, we may assume $A(p_1p_2p_3) \geq \frac{5-\sqrt{5}}{10}$. Then we can take point q on p_2p_3 such that $A(p_1p_2q) = \frac{\tau}{\tau+2}$. By induction, the pentagon $p_1qp_3p_4p_5$ contains internally disjoint triangles T_1 and T_2 such that $A(T_1) = A(T_2) = \frac{\tau}{2} \times A(p_1qp_3p_4p_5) = \frac{\tau}{\tau+2} = A(p_1p_2q)$. Thus $t_3(K) \geq \frac{3\tau}{\tau+2}$. Since $\tau \geq \frac{2}{3}$ by (i), $t_3(K) \geq 3\left(1 - \frac{2}{\tau+2}\right) \geq \frac{3}{4}$, as desired.

(iii) We may assume that $A(K) = 1$ and $A(p_1p_2p_3) \geq \frac{5-\sqrt{5}}{10}$ (recall Proposition 3). The proof is by induction on n. We first show that $t_4(K) \geq \frac{8}{9}$. By Proposition 4, we can divide K into two convex polygons Q_1 and Q_2 each with at most four sides and $A(Q_1) = A(Q_2) = \frac{1}{2}$. Hence by Theorem 2, we can take internally disjoint triangles $T_1, T_2 \subset Q_1$ and $T_3, T_4 \subset Q_2$ such that $A(T_1) = \cdots = A(T_4) = \frac{4}{9} \times \frac{1}{2} = \frac{2}{9}$. Thus $t_4(K) \geq \frac{8}{9}$. Next suppose that $t_n(K) \geq \frac{2n}{2n+1}$ for some $n \geq 4$. Take point q on p_2p_3 such that $A(p_1p_2q) = \frac{2}{2(n+1)+1} \left(< \frac{5-\sqrt{5}}{10}\right)$. By our induction hypothesis, the pentagon $p_1qp_3p_4p_5$ contains n internally disjoint triangles $T_1, \cdots, T_n$ such that $A(T_1) = \cdots = A(T_n) = \frac{2}{2n+1} \times A(p_1qp_3p_4p_5) = \frac{2}{2(n+1)+1} = A(p_1p_2q)$. Consequently, $t_{n+1}(K) \geq \frac{2(n+1)}{2(n+1)+1}$, as desired. □

For a positive integer n and a regular hexagon K, we have by Theorem D that $t_{6n}(K) = 1$. We show here that:

Theorem 4. *Let $n \geq 2$ be an integer and let K be a convex polygon with at most six sides. Then $t_{3n}(K) \geq \frac{4n}{4n+1}$.*

Proof. We may assume that $A(K) = 6$. We first show that K can be divided into two polygons K_1 and K_2 such that K_1 has at most four sides, $A(K_1) = 2$, K_2 has at most five sides and $A(K_2) = 4$. Let $K = p_1p_2 \cdots p_6$. In the case where K has $k < 6$ sides, take $6 - k$ points on one of its edges, and think of them as $6 - k$ artificial vertices of K which can now be considered as a convex hexagon. Write $T_1 = p_1p_2p_3$, $T_2 = p_3p_4p_5$, $T_3 = p_5p_6p_1$. We may assume that $A(T_1) < 2$. First consider the case where $A(T_3) < 2$. In this case, we may assume further that $A(p_1p_2p_3p_4) \geq 3$ by symmetry. Then there exists a point $q \in p_3p_4$ such that $A(p_1p_2p_3q) = 2$ and $A(p_1qp_4p_5p_6) = 4$, as desired. Thus consider the case where $A(T_3) \geq 2$. Since $A(T_1) < 2$ and $A(p_1p_2p_3p_5p_6) > A(T_3) \geq 2$, either there exists a point $q \in p_6p_1$ such that $A(p_1p_2p_3q) = 2$, or there exists a point $q \in p_5p_6$ such that $A(p_1p_2p_3qp_6) = 2$. In the former case, $K_1 = p_1p_2p_3q$ and $K_2 = p_3p_4p_5p_6q$ have the desired properties. In the latter case, since $A(p_1p_2qp_6) < A(p_1p_2p_3qp_6) = 2$ and $A(p_1p_2p_5p_6) > A(T_3) \geq 2$, there exists a point $r \in qp_5$ such that $A(p_1p_2rp_6) = 2$ and $A(p_2p_3p_4p_5r) = 4$, as desired.

Now since $t_n(K_1) \geq \frac{4n}{4n+1}$ by Theorem 2 (ii) and $t_{2n}(K_2) \geq \frac{4n}{4n+1}$ by Theorem 3 (iii), we obtain $t_{3n}(K) \geq \frac{4n}{4n+1}$, as desired. □

4 Equal Area Polygons in a Convex Body

Let K be a convex body in $\boldsymbol{E}^2$. Combining Theorem B and Proposition 4, we obtain several results. For example, for any integer $n \geq 3$,

$$\tfrac{2n-3}{2\pi} \sin \tfrac{2\pi}{2n-3} \leq s(K;1,2n-3) \leq s(K;2,n). \tag{10}$$

Similarly, for $m = 2^l$, $l = 0,1,2,\cdots$, we have $s(K;m,4) \geq s(K;\frac{m}{2},5) \geq \cdots \geq s(K;1,m+3) \geq \frac{m+3}{2\pi} \sin \frac{2\pi}{m+3}$, and hence

$$s(K;2m,3) \geq \tfrac{8}{9} s(K;m,4) \geq \tfrac{4(m+3)}{9\pi} \sin \tfrac{2\pi}{m+3} \tag{11}$$

by Theorem 2. On the other hand, it follows from Proposition 2 that

$$s(K;2,n) \leq s(K;1,2n-2). \tag{12}$$

We henceforth focus on $s(K;2,n)$. By (10) and (12),

$$s(K;1,2n-3) \leq s(K;2,n) \leq s(K;1,2n-2) \text{ for } n \geq 3, \tag{13}$$

and by (11) and (12),

$$\tfrac{8}{9} s(K;1,4) \leq s(K;2,3) \leq s(K;1,4). \tag{14}$$

We believe that the following is true:

Conjecture 1. Let K be a convex body in $\boldsymbol{E}^2$. Then $s(K;2,n) \geq \frac{n-1}{\pi} \sin \frac{\pi}{n-1}$ with equality if and only if K is an ellipse.

Remark 3. We can verify that the equality of this conjecture holds if K is an ellipse in the following way: Let E be an ellipse. Since a circular disk D contains a regular $2(n-1)$-gon R with $A(R) = \frac{n-1}{\pi} \sin \frac{\pi}{n-1} A(D)$, E contains a centrally symmetric $2(n-1)$-gon P with $A(P) = \frac{n-1}{\pi} \sin \frac{\pi}{n-1} A(E)$, which can be divided into two internally disjoint equal area n-gons. Thus $s(E;2,n) \geq \frac{n-1}{\pi} \sin \frac{\pi}{n-1}$. Furthermore, we have $s(E;2,n) \leq s(E;1,2n-2) = \frac{n-1}{\pi} \sin \frac{\pi}{n-1}$ by (12) and Theorem B. Consequently, $s(E;2,n) = \frac{n-1}{\pi} \sin \frac{\pi}{n-1}$ holds for any ellipse E.

In this section, we settle Conjecture 1 affirmatively for some special cases.

Theorem 5. *Let K be a centrally symmetric convex body in $\boldsymbol{E}^2$. Then $s(K;2,n) \geq \frac{n-1}{\pi} \sin \frac{\pi}{n-1}$.*

To prove Theorem 5, it suffices to show the following:

$$\left.\begin{array}{l}\text{Let } K \text{ be a centrally symmetric convex body in } \boldsymbol{E}^2\text{. Then there exists}\\ \text{a polygon } P \subseteq K \text{ with } \frac{A(P)}{A(K)} \geq \frac{n-1}{\pi} \sin \frac{\pi}{n-1} \text{ such that } P \text{ has at most}\\ 2n-2 \text{ sides and } P \text{ is centrally symmetric with respect to the center}\\ \text{of } K.\end{array}\right\} \tag{15}$$

Observe that P can then be divided into two internally disjoint equal area polygons with at most n sides. We show (15) in a generalized form stated in the following Theorem 6.

Let n be a positive integer. For a subset S of $\boldsymbol{E}^n$ having a finite volume, let $V(S)$ denote the volume of S. For a centrally symmetric convex body K in $\boldsymbol{E}^n$, denote by $\mathcal{Q}_m(K)$ the set of polytopes P contained in K such that P is centrally symmetric with respect to the center of K and P has at most $2m$ vertices. Let

$$\sigma(K;m) = \sup_{P\in\mathcal{Q}_m(K)} \frac{V(P)}{V(K)}.$$

Theorem 6. *Let m and n be integers with $m \geq n \geq 2$. Let K be a centrally symmetric convex body in $\boldsymbol{E}^n$ and let S be a hyper-sphere in $\boldsymbol{E}^n$. Then $\sigma(K;m) \geq \sigma(S;m)$.*

Proof. Our proof is a modification of the proof of the n-dimensional theorem of Theorem B by Macbeath [8], where *Steiner symmetrization* is applied. We give only a sketch of our proof.

We may assume that K is centrally symmetric with respect to the origin O of $\boldsymbol{E}^n$. Let π be a hyper-plane in $\boldsymbol{E}^n$ containing the origin O. Denote each point a in $\boldsymbol{E}^n$ by (x,t), where $x = x(a)$ is the foot of the perpendicular from a to π and $t = t(a)$ is the oriented perpendicular distance from x to a. For a convex body B, let B' be the projection of B on π. For $x \in B'$, define the two functions $B^+(x)$ and $B^-(x)$ by $B^+(x) = \sup_{(x,t)\in B} t$ and $B^-(x) = \inf_{(x,t)\in B} t$. Then $B = \{\,(x,\,t)\,|\,x\in B',\ B^-(x)\leq t\leq B^+(x)\,\}$.

Let $K^* = \{\,(x,\,t)\,|\,x \in K',\ |t| \leq \frac{1}{2}(K^+(x)-K^-(x))\,\}$. Then K^* is symmetric with respect to π, centrally symmetric with respect to O, and $V(K^*) = \int_{K'}(K^+(x)-K^-(x))dx = V(K)$. By the central symmetry of K with respect to O,

$$-x\in K',\ K^+(-x) = -K^-(x) \text{ and } K^-(-x) = -K^+(x) \text{ for any } x\in K'. \quad (16)$$

Lemma 1. $\sigma(K^*;m) \leq \sigma(K;m)$

Proof. Let P be a polytope in $\mathcal{Q}_m(K^*)$. It suffices to show that there is a polytope $P_0 \in \mathcal{Q}_m(K)$ such that $V(P_0) \geq V(P)$. Let $2k\,(\leq 2m)$ be the number of vertices of P and let $(x_i,\,t_i), 1 \leq i \leq 2k$, be the vertices of P. We label the indices so that for each $1 \leq i \leq k$, $(x_i,\,t_i)$ and $(x_{k+i},\,t_{k+i})$ are symmetric with respect to O (so $(x_{k+i},\,t_{k+i}) = (-x_i,\,-t_i)$). Let Q be the convex hull of the points $(x_i,\,t_i+\frac{1}{2}(K^+(x_i)+K^-(x_i)))$, $1\leq i\leq 2k$, and let R be the convex hull of the points $(x_i,\,-t_i+\frac{1}{2}(K^+(x_i)+K^-(x_i)))$, $1 \leq i \leq 2k$. Since $|t_i| \leq \frac{1}{2}(K^+(x_i)-K^-(x_i))$, each vertex of Q and R is contained in K, and hence $Q,\,R \subseteq K$. Also, since for each $1 \leq i \leq k$,

$$\tfrac{1}{2}(x_i+x_{k+i}) = \tfrac{1}{2}(x_i+(-x_i)) = 0 \quad \text{and}$$
$$\tfrac{1}{2}\left[t_i+\tfrac{1}{2}(K^+(x_i)+K^-(x_i))+t_{k+i}+\tfrac{1}{2}(K^+(x_{k+i})+K^-(x_{k+i}))\right]$$

$$
\begin{aligned}
&= \tfrac{1}{2}\left[t_i + \tfrac{1}{2}(K^+(x_i) + K^-(x_i)) + (-t_i) + \tfrac{1}{2}(K^+(-x_i) + K^-(-x_i))\right] \\
&= \tfrac{1}{2}\left[\tfrac{1}{2}(K^+(x_i) + K^-(-x_i)) + \tfrac{1}{2}(K^+(-x_i) + K^-(x_i))\right] \\
&= 0
\end{aligned}
$$

by (16), Q is centrally symmetric with respect to O. Similarly, we see that R is centrally symmetric with respect to O. Furthermore, since

$$
\begin{aligned}
Q^-(x_i) \ &\le\ t_i + \tfrac{1}{2}(K^+(x_i) + K^-(x_i)) \ \le\ Q^+(x_i) \quad \text{and} \\
R^-(x_i) \le &\ -t_i + \tfrac{1}{2}(K^+(x_i) + K^-(x_i)) \le\ R^+(x_i),
\end{aligned}
$$

we have that $\frac{1}{2}(Q^-(x_i) - R^+(x_i)) \le t_i \le \frac{1}{2}(Q^+(x_i) - R^-(x_i))$, and hence each point $(x_i,\, t_i)$, $1 \le i \le 2k$, lies in the convex set

$$T = \{\, (x,\, t) \,|\, x \in P',\ \tfrac{1}{2}(Q^-(x) - R^+(x)) \le t \le \tfrac{1}{2}(Q^+(x) - R^-(x)) \,\}.$$

Since P is the convex hull of the points $(x_i,\, t_i)$, $1 \le i \le 2k$,

$$
\begin{aligned}
V(P) \ \le\ V(T) &= \tfrac{1}{2}\textstyle\int_{P'} (Q^+(x) - Q^-(x) + R^+(x) - R^-(x))\, dx \\
&= \tfrac{1}{2}(V(Q) + V(R)).
\end{aligned}
$$

Thus at least one of $V(Q) \ge V(P)$ or $V(R) \ge V(P)$ holds. Consequently, Q or R is a polytope with desired properties. □

Now we return to the proof of Theorem 6. The rest of our argument follows exactly as the proof in [8]: we can verify that $\sigma(K; m)$ is a continuous function of K. Let $\pi_1, \pi_2, \cdots, \pi_n$ be n hyper-planes such that for each pair $i \ne j$ π_i and π_j form an angle which is an irrational multiple of π. Consider the sequence of bodies $K = K_1, K_2, \cdots, K_n, \cdots$, where K_{i+1} arises from K_i by symmetrizing it with respect to π_ν where ν is the least positive residue of i (mod n). This sequence converges to a hyper-sphere S (see [3]), and hence $\sigma(K; m) \ge \sigma(S; m)$. □

Let K be a convex body in $\boldsymbol{E}^2$ and let l denote the perimeter of K. Then

$$\textit{The Isoperimetric Inequality:}\quad l^2 \ge 4\pi A(K) \tag{17}$$

with equality if and only if K is a circular disk; and, if K is a figure with constant width w, we also have

$$\textit{Barbier's Theorem:}\quad l = \pi w \tag{18}$$

(see, for example, [4]). Finally we show that Conjecture 1 is true for $n = 3$ when K is a figure with constant width:

Theorem 7. *Let K be a figure with constant width in $\boldsymbol{E}^2$. Then $t_2(K) \ge \frac{2}{\pi}$ with equality if and only if K is a circular disk.*

Proof. Let w and l denote the width and perimeter of K, respectively. For each $\theta \in [0, 2\pi)$, let $\boldsymbol{u}_\theta$ denote the vector $(\cos\theta, \sin\theta)$, let $a = a_\theta$ and $b = b_\theta$ denote the contact points of K and each of two straight lines parallel to $\boldsymbol{u}_\theta$, and let

$m = m_\theta$ denote the midpoint of the segment ab (Fig. 8 (a)). Let $c = c_\theta$ and $d = d_\theta$ be the intersection points of the perimeter of K and the straight line passing through m and parallel to $\boldsymbol{u}_\theta$. Then we have $A(acd) = A(bcd)$. Take c' on the line tangent to the perimeter of K at c such that $\det \left[\overrightarrow{cc'}\ \boldsymbol{u}_\theta\right] > 0$, where $\left[\overrightarrow{cc'}\ \boldsymbol{u}_\theta\right]$ stands for a matrix having $\overrightarrow{cc'}$ and $\boldsymbol{u}_\theta$ as their column vectors. We further take d' on the tangent line of the perimeter of K at d such that $\det \left[\overrightarrow{dd'}\ \boldsymbol{u}_\theta\right] < 0$. Write $\alpha_1 = \alpha_1(\theta) = \angle mcc'$ and $\alpha_2 = \alpha_2(\theta) = \angle mdd'$.

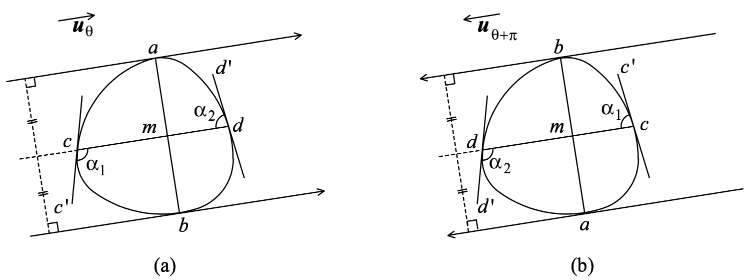

Fig. 8.

Since $\alpha_1(\theta + \pi) - \alpha_2(\theta + \pi) = -(\alpha_1(\theta) - \alpha_2(\theta))$ (Fig. 8 (a),(b)), it follows from the Intermediate Value Theorem that there exists $\theta \in [0, \pi]$ such that $\alpha_1(\theta) - \alpha_2(\theta) = 0$ i.e. $cc' \parallel dd'$. For this θ, we have $cd \geq w$, and hence

$$A(acd) + A(bcd) = \frac{1}{2} cd \cdot w \geq \frac{1}{2} w^2 = \frac{1}{2}\left(\frac{l}{\pi}\right)^2 \geq \frac{2}{\pi} \cdot A(K)$$

by (17) and (18). Furthermore, if $t_2(K) = \frac{\pi}{2}$ holds, then we must have $l^2 = 4\pi A(K)$, i.e., K is a circular disk; and for a circular disk K, we have $t_2(K) = \frac{\pi}{2}$ (recall Remark 3). □

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