*

Illumination of Polygons with Vertex Lights

Vladimir Estivill-Castro [a], Joseph O'Rourke [b,1] Jorge Urrutia [c]
Dianna Xu [d]

[a] *Laboratorio Nácional de Informática Avanzada, Rébsamen 80, Xalapa, Veracruz 91000, México. vlad@xalapa.lania.mx.*

[b] *Dept. of Computer Science, Smith College, Northampton, MA 01063, USA. orourke@cs.smith.edu.*

[c] *Dept. of Computer Science, University of Ottawa, Ottawa, Ontario, Canada K1N 6N5. jorge@csi.uottawa.ca.*

[d] *Dept. of Computer Science, Smith College. dxu@cs.smith.edu.*

Abstract

We show that vertex floodlights of angle π suffice to illuminate any polygon, and that no angle less than π suffices for all polygons. A *vertex floodlight* of angle α is an internal cone of light of aperture no more than α with apex at a vertex. No vertex may have more than one floodlight.

Keywords: Computational geometry, visibility.

1 Introduction

There has been recent interest in illumination by floodlights [1] [2] [6], a new variation on visibility problems in computational geometry. A floodlight of angle α is a light that projects in a (contiguous) cone of angle at most α. A floodlight placed in a simple polygon with its apex at a vertex, aimed entirely interior to the polygon, is called a *vertex floodlight*; if it has angle at most α, we will call it a *vertex α-light*. We restrict each vertex to anchoring at most one floodlight. The question of illuminating a polygon with vertex floodlights was first explored in [3], in which it was established that every orthogonal polygon can be covered by $\lfloor 3(n-1)/8 \rfloor$ vertex $\pi/2$-lights. This work raised

[1] Supported by NSF grants CCR-9122169 and CCR-9421670.

the question of what angle of vertex floodlights would suffice to cover any simple polygon, permitting n floodlights, one per vertex. In [5] it was shown that vertex $\pi/2$-lights do not suffice, even for monotone polygons. In this note we show that vertex π-lights suffice, and no smaller angle suffices for all polygons. [2] Henceforth we sometimes shorten "floodlight" to "light" and "vertex α-light" to "α-light."

2 π Suffices

The proof is by induction on the number of vertices of P. If P is a triangle, a floodlight at any vertex suffices to cover it. Suppose P has $n > 3$ vertices. P must have an ear, by Meister's Two Ears Theorem [4, p .16]: three consecutive vertices a, b, c such that ac is an internal diagonal. The angle at the ear tip vertex b is strictly convex. Place a floodlight to cover the interior angle at b ($< \pi$) and cover the ear triangle (a, b, c). Removing the ear leaves a polygon of $n - 1$ vertices, which can be covered by the induction hypothesis.

3 $\pi/2$ Does Not Suffice

Although it is already known that $\pi/2$-lights do not suffice, the polygons we use later employ variations on the mechanism introduced in [5] to establish this result. We reestablish and simplify this result with a construction that will form the basis of modifications in the next section.

The polygon P is shown in Fig. 1. It is symmetric about a vertical line through

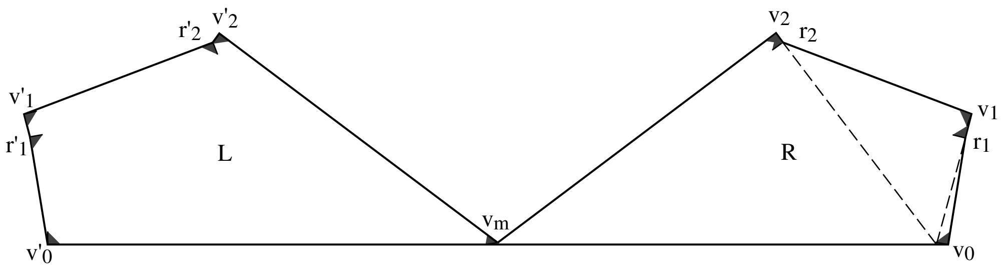

Fig. 1. A polygon for which vertex $\pi/2$-lights do not suffice.

the middle vertex v_m. Let L be the half of P left of the vertical through v_m, and let R be the right half. Vertices in L are labeled to reflect their counterparts in R. The internal angles at vertices v_0, v_1, and v_2 all exceed $\pi/2$; vertices r_1,

[2] A preliminary version of this work was presented at the *4th MSI Workshop on Comp. Geom.*, Cornell University, Oct. 14-15, 1994.

r_2, and v_m are reflex. The extensions of the edges r_1v_1 and of r_2v_2 (dashed in the figure) intersect the horizontal edge v_0v_0' slightly left of v_0. Vertex v_m is just above the edge v_0v_0'. Floodlights are shown at each vertex by a small shaded $\pi/2$-wedge (the one at v_m could be truncated to be entirely interior).

We now argue that $\pi/2$-lights cannot cover P. The light at v_m may illuminate points in either L or R but not both. Assume without loss of generality that this light aims leftward as shown in Fig. 1. Because the angles at v_i exceed $\pi/2$, the lights at these vertices must leave points on the boundary ∂P of P near v_i locally unilluminated. Let x_i^- and x_i^+ be distinguished points on ∂P very close to v_i, with x_i^- clockwise (cw) and x_i^+ counterclockwise (ccw) of v_i in a boundary traversal; call both points "x_i points." There are two primary choices for each v_i light: turn cw as far as possible, in which case the light covers x_i^+ but not x_i^- (Fig. 2a), or turn ccw as far as possible, in which case the light covers x_i^- but not x_i^+. Any intermediate orientation of the light will not cover either x_i point.

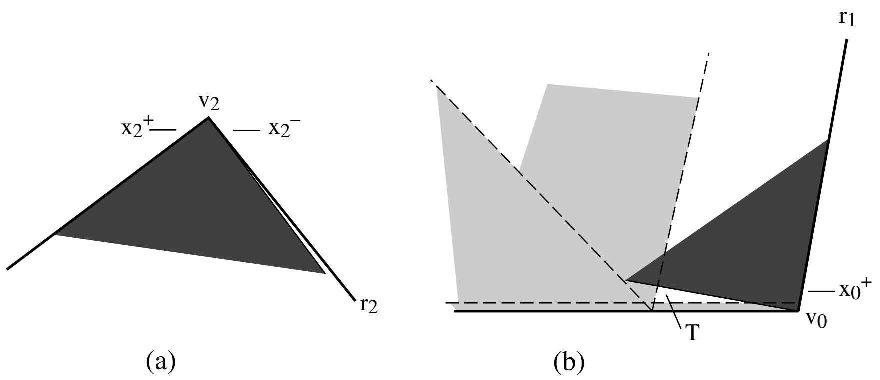

Fig. 2. An enlargement of the vicinities of v_2 (a) and v_0 (b) in Fig. 1.

The x_2 points are visible from only three vertices: v_m, v_2, and r_2 (v_0 cannot see v_2 because it is right of the extension of r_2v_2). The light at v_m is aiming into L and of no use, so these points must be covered from v_2 and r_2. Since the angle at v_2 exceeds $\pi/2$, the light at v_2 can cover at most one x_2 point. So the light at r_2 must be used by turning it cw, in which case it covers both x_2 points.

Now consider the two x_1 points close to v_1: they are visible from the vertices v_m, r_2, v_1, and r_1. The lights at v_m and r_2 are aimed away and can cover neither. The light at v_1 can cover at most one of the x_1 points. Therefore the light at r_1 must be used by turning it cw, thereby covering both x_1 points.

If the light at v_0 is turned ccw flush with v_0v_0', a point x_0^+ on v_0r_1 above the $v_0'v_m$ line is uncovered, since the light at r_1 is aimed away. Therefore the light at v_0 must be turned cw flush with v_0r_1, as shown in Figs. 1 and 2b. We will see in a moment that this still leaves points near v_0 uncovered.

Now we turn attention to the left half of P. Because the light at v_m shines into L, seeing both v_1' and v_2', the lights at the other vertices of L may be aimed in any direction, leaving all but a small triangle of L under v_m covered. In Fig. 1 the cw orientation is shown for every light. This arrangement has the light at v_0' flush with $v_0'v_0$, and therefore shining down that edge. This allows a thin beam of light to shine into R from v_0'.

Finally, we reexamine the vicinity of v_0, shown magnified in Fig. 2b. Although all the points on ∂P in a small neighborhood of v_0 are illuminated, there is a small triangle T of interior points that remain unilluminated. The light from v_0' to a point in T is blocked by v_m, from v_1 it is blocked by r_1, and from v_2 it is blocked by r_2. Thus P cannot be covered with $\pi/2$-lights.

4 $\pi - \epsilon$ Does Not Suffice

We now generalize the design in the preceeding section to show that any angle $\alpha < \pi$ does not suffice. We first describe the polygon for a particular α, and second discuss the general construction.

4.1 A Particular Polygon

The polygon P depicted in Fig. 3 is a direct generalization of Fig. 1, again symmetrical with respect to a vertical line through v_m. The right half is composed of vertices $v_0, r_1, v_1, \ldots, v_k, r_k$. Vertices $v_0, \ldots, v_k$ all have internal angles exactly equal to α. In the previous section the angles at v_i were greater than $\pi/2$ and we argued that $\pi/2$-lights do not suffice; here it is more convenient to set the angles at v_i equal to α and argue that no smaller angle suffices. Fig. 3 shows a portion of the right half for $k = 5$, $\alpha = 3\pi/4 = 135°$. The edge $v_k v_m$ slopes downward to the left, and (as in Fig. 1) v_m is just above the $v_0 v_0'$ edge.

The vertices r_i, $i = 0, \ldots, k$ are placed very near to v_i, (so near as to be indiscernable in Fig. 3), and are slightly reflex. A magnified view of the vicinity of v_3 is shown in the circle in Fig. 3. The edges $v_0 r_1, v_1 r_2, v_2 r_3, \ldots$ increase in length in a manner that will be detailed later. The extensions of the edges $r_i v_i$ (dashed in the figure) intersect $v_0 v_0'$ slightly left of v_0. For concreteness we arrange that they all meet $v_0 v_0'$ at the same point x (so close to v_0 as to be indistinguishable in Fig. 3).

We now argue that that P cannot be covered by an angle $\alpha' < \alpha$. The proof is exactly analogous to that in the previous section. Assume without loss of generality that the light at v_m aims into L. Then two distinguished points on

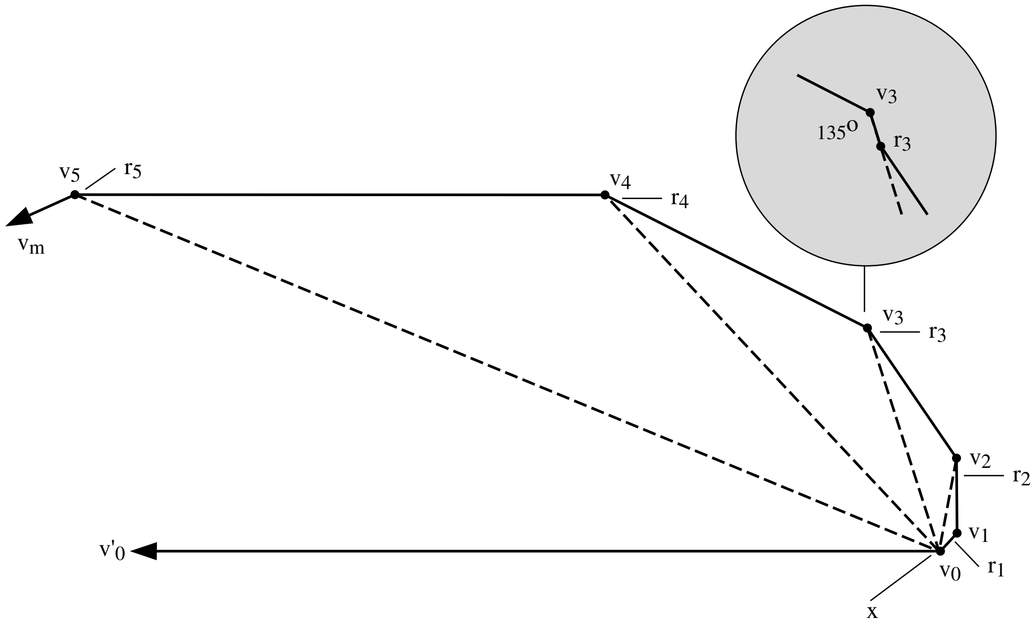

Fig. 3. Vertex $(135° - \epsilon)$-lights do not suffice. Only a portion of the right half R of P is shown. The vicinity of v_3 is magnified in the circle.

∂P in a small neighborhood and to either side of v_k can only be illuminated from v_k and r_k, since all of the vertices with indices $k-1, \ldots, 0$ lie right of the extension of $r_k v_k$. Because the light at v_k cannot cover both points, the light at r_k must be turned cw. Now the argument is repeated at $v_{k-1}, \ldots, v_1$, concluding that the lights at r_i must all be turned cw. Finally this requires that the light at v_0 be turned cw to cover a point x_0^+ on $v_0 r_1$. Just as in the previous section, this leaves a small interior region of points (analogous to triangle T in Fig. 2b) unilluminated, regardless of how the left half L is covered.

4.2 General Construction

For arbitary $\alpha < \pi$, the main challenge is to arrange that the extensions of the edges $r_i v_i$ all intersect $v_0 v_0'$ at point x left of v_0, while maintaining the property that each angle at v_i is α. One can base a general construction on the properties of the logarithmic spiral, $r = ae^{b\theta}$. This spiral has the property that the angle between the radial vector to a point p on the curve and the tangent at p is a constant. If $b = \cot \alpha$, this constant angle is α.

Now we show how to use the family of spirals $r = ae^{(\cot \alpha)\theta}$, $a = 1, 2, \ldots$ to construct the polygon P. Call these spirals $\mathcal{S}_a$. We illustrate the technique for $\alpha = 3\pi/4 = 135°$ in Fig. 4. The origin of the spirals becomes x, the point just left of v_0 in P. Draw an edge from v_0 (just right of x) to $\mathcal{S}_1$ forming an angle α with the horizontal; the intersection point becomes v_1. From v_1 follow

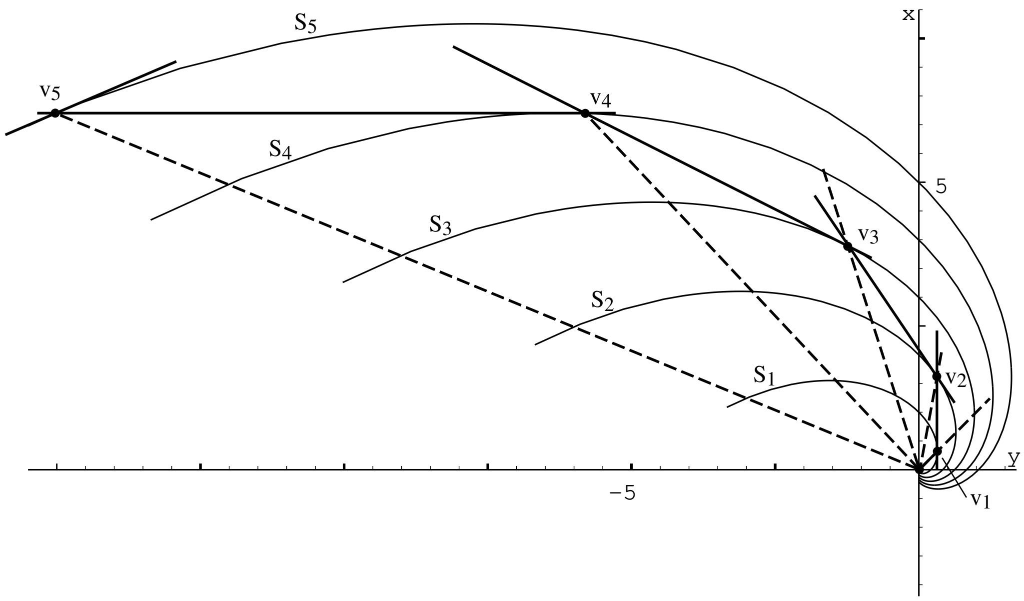

Fig. 4. Logarithmic spiral construction for $\alpha = 3\pi/4 = 135°$. the origin is x, and v_0 is imperceptibly right of x. Note the positive x-axis is vertical in this figure.

the tangent line of $\mathcal{S}_1$ to $\mathcal{S}_2$; this intersection is v_2. This process is continued until the tangent line slopes downward. In Fig. 4, this is achieved at v_5. (The tangent is horizontal [parallel to the y-axis] when $\theta = -(\alpha - \pi/2)$, a negative angle measured from the vertical positive x-axis; this occurs at v_4 in the figure, when $\theta = -\pi/4 = -45°$.)

By the properties of the logarithmic spiral, we are guaranteed that the angle between xv_i (the radial vector) and $v_i v_{i+1}$ (the tangent vector) is α. Because $\alpha < \pi$, we are guaranteed that θ advances on the spirals from v_i to v_{i+1}. It can be shown that the advance exceeds $(\pi - \alpha)/2$ as $\alpha \to \pi$; in Fig. 4, the advance exceeds $\pi/8 = 22.5°$. This advancement in turn guarantees that for some finite k, the tangent will slope downwards.[3] With this basic structure established, it is easy to insert the reflex vertices r_i close to v_i by a slight expansion along the xv_i line (as illustrated for v_3 in Fig. 3). The result is a polygon P for which floodlights of angle smaller than α do not suffice.

This construction works for any $\alpha < \pi$. (Note that the logarithmic spiral is not defined for $\alpha = \pi$, when $\cot \alpha = -\infty$.) Thus, given any angle $\alpha < \pi$, we can construct a polygon that cannot be illuminated by vertex lights of angle α' smaller than α. So for any $\epsilon > 0$, no angle $\pi - \epsilon$ suffices for coverage: choose $\alpha = \pi - \epsilon/2$.

[3] An argument that we will not detail shows that $k < 2\pi/(\pi - \alpha)$.

5 Discussion

Now that we know that no angle less than π suffices, it becomes an interesting question to ask how many vertex π-lights are ever needed as a function of n, the number of vertices of P.

References

[1] P. Bose, L. Guibas, A. Lubiw, M. Overmars, D. Souvaine, and J. Urrutia. The floodlight problem. In *Proc. 5th Canad. Conf. Comput. Geom.*, pages 399–404, Waterloo, Canada, 1993.

[2] J. Czyzowicz, E. Rivera-Campo, and J. Urrutia. Optimal floodlight illumination in polynomial time. In *Proc. 5th Canad. Conf. Comput. Geom.*, pages 393–398, Waterloo, Canada, 1993.

[3] V. Estivill-Castro and J. Urrutia. Optimal floodlight illumination of orthogonal art galleries. In *Proc. 6th Canad. Conf. Comput. Geom.*, pages 81–86, 1994.

[4] J. O'Rourke. *Computational Geometry in C*. Cambridge University Press, 1994.

[5] J. O'Rourke and D. Xu. Illumination of polygons with 90° vertex lights. In *Snapshots in Comput. Geom.*, pages 108–117. Univ. Saskatchewan, August 1994.

[6] W. Steiger and I. Streinu. Positive and negative results on the floodlight problem. In *Proc. 6th Canad. Conf. Comput. Geom.*, pages 87–92, 1994.