On the $\mathcal{O}$-hull of planar point sets

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Abstract

Let P be a set of n points in the plane and $\mathcal{O}$ be a set of k, $2 \leq k \leq n$, different orientations in the plane sorted in counterclockwise circular order, such that the biggest angle defined by two consecutive orientations is at most $\frac{\pi}{2}$. We show: (1) How to compute the oriented $\mathcal{O}$-hull of P in optimal $\Theta(n \log n)$ time and $O(n)$ space, (2) how to compute the unoriented $\mathcal{O}$-hull of P in $O(kn \log n)$ time and $O(kn)$ space, and (3) how to solve the problem of computing an orientation of the plane for which the $\mathcal{O}$-hull of P has minimum area in $O(kn \log n)$ time and $O(kn)$ space.

1 Introduction

Let P be a set of n points in the plane, in general position. In [1] the authors solved the problem of computing the orientation of the plane such that the rectilinear convex hull of P, denoted by $\mathcal{RH}(P)$, has minimum area (or perimeter) in optimal $\Theta(n \log n)$ time and $O(n)$ space.

In this paper we will extend those results for $\mathcal{RH}(P)$ to the $\mathcal{O}$-hull of the set P, denoted by $\mathcal{OH}(P)$, where the horizontal and vertical lines are replaced by a set $\mathcal{O}$ of k different lines, with $2 \leq k < n$. Due to the lack of space, we will omit some of the details.

1.1 $\mathcal{O}$-convexity

A *set of orientations* $\mathcal{O}$ is given by a set of k lines, passing through a fixed point and such that each line defines the two corresponding orientations. Thus, in fact, we have $2k$ different orientations in the inter-

val $[0, 2\pi)$. In this work we assume that the largest angle between two consecutive orientations is at most $\frac{\pi}{2}$.

If an $\mathcal{O}$-*line* is a translation of an element of $\mathcal{O}$, a region R in the plane is said to be $\mathcal{O}$-*convex* if the intersection of R with any $\mathcal{O}$-line is either empty or connected. This notion, introduced by Rawlins [11] to generalize the orthoconvexity studied by Ottmann et al. [9], was later considered by Rawlins and Wood [12], Martynchik et al. [7], and Fink and Wood [5]. Also by Matoušek et al. [6, 8], whose definitions seem not to be completely equivalent.

1.2 $\mathcal{O}$-hull of a planar point set

The usual convexity, defined requiring the intersection with any line of the plane to be either empty or connected, leads to a number of equivalent definitions of the usual convex hull of a point set, $\mathcal{CH}(P)$. On the contrary, Ottmann et al. [9] observed that three different definitions (r-, cr-, and mr-hull) can be considered for the orthoconvex hull of a point set.

Here we will use the following definition for the $\mathcal{O}$-hull of P.

Definition 1 $\mathcal{OH}(P)$ *is the intersection of all the connected supersets of P which are $\mathcal{O}$-convex (see Figure 1).*

(a) Set $\mathcal{O}$ with $k = 3$.
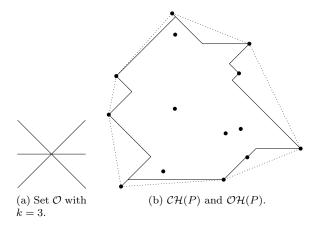
(b) $\mathcal{CH}(P)$ and $\mathcal{OH}(P)$.

Figure 1: Example of $\mathcal{O}$ and $\mathcal{OH}(P)$.

First, we characterize this definition in terms of wedges, showing its correspondence to the mr-hull used in [1, 3]. This will allow us to use the techniques based on Θ-maximal elements of P (see Avis et al. [2]) for computing the unoriented $\mathcal{OH}(P)$, in a similar way as for $\mathcal{RH}(P)$ in [1].

Consider two consecutive orientations $o_2 \succ o_1$ in the circular counterclockwise order of $\mathcal{O}$. Assume that P is located on the intersection of the two right half-planes defined by o_1 and o_2. When sweeping to the right the lines supporting these orientations, we want the intersection of $\mathcal{OH}(P)$ with any o_1-line to be connected, as well as the intersection with any o_2-line.

Suppose that during this sweeping the o_1-line bumps a first point $p \in P$ and the o_2-line bumps a first point $q \in P$, and $p \neq q$. After that, the process of maintaining connected the intersection of $\mathcal{OH}(P)$ with both sweeping oriented lines implies that the "boundary limit" of $\mathcal{OH}(P)$ between p and q has to be formed by a staircase polygonal chain with alternating constant turn angle $\pi - (o_2 - o_1)$. This motivates the following definition.

Definition 2 *Let o_1 and o_2 be two consecutive orientations in $\mathcal{O}$. The stabbing wedge associated to $\angle o_1 o_2$, denoted by $w_{o_1 o_2}$ or just w when clear enough, is defined by rays parallel to $-o_1$ and o_2, hence having size $\pi - (o_2 - o_1)$. In other words, it is the wedge supplementary to $\angle o_1 o_2$. See Figure 2.*

We call *stabbing $\mathcal{O}$-wedges* those associated to pairs of consecutive orientations in $\mathcal{O}$. By our assumption, all the angles of the stabbing wedges are at least $\frac{\pi}{2}$.
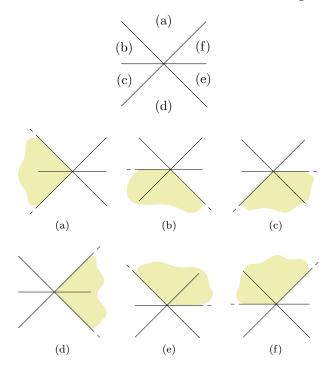

Figure 2: Top: Pairs of consecutive orientations in $\mathcal{O}$. Bottom: Associated stabbing wedges.

In the case $p = q$, the corresponding part of the boundary of $\mathcal{OH}(P)$ is just the point p. This happens for the topmost point in Figure 1b with the directions o_1 and o_2, so that the stabbing wedge (f) in Figure 2 is not acting. The above process leads to this result:

Proposition 1 *The $\mathcal{O}$-convex hull of P, $\mathcal{OH}(P)$, is*

$$\mathcal{OH}(P) = \mathbb{R}^2 - \bigcup_{w \in \mathcal{W}} w,$$

where $\mathcal{W}$ is the set of all stabbing $\mathcal{O}$-wedges free of elements of P.

2 Computing the oriented and unoriented $\mathcal{OH}(P)$

From Proposition 1 we get an algorithmic method to compute $\mathcal{OH}(P)$, by computing all the $\mathcal{O}$-wedges free of elements of P. Let h be the number of edges (or vertices) of $\mathcal{CH}(P)$, let $V = \{p_0, \ldots, p_{h-1}\}$ be the vertex set in the boundary of $\mathcal{CH}(P)$ sorted counterclockwise, and let $E = \{e_0, \ldots, e_{h-1}\}$ be the edge set, where $e_i = \overline{p_i p_{i+1}}$ and the addition in the sub-indices is taken modulo h.

2.1 Computing the oriented $\mathcal{OH}(P)$

To see where a stabbing $\mathcal{O}$-wedge $w_{o_1 o_2}$ can penetrate in $\mathcal{CH}(P)$, we compute the supporting directed line of $\mathcal{CH}(P)$ parallel to o_1 such that $\mathcal{CH}(P)$ is on its right side and look for the tangent point p_i of this supporting line. We also compute the tangent point p_j of the supporting directed line of $\mathcal{CH}(P)$ parallel to o_2. Since $o_2 \succ o_1$, we have that $p_j \succ p_i$ in the counterclockwise order of V. The *stabbing interval* of the stabbing $\mathcal{O}$-wedge $w_{o_1 o_2}$ is then the interval $[i, j]$, $0 \le i, j < h$. Figure 3 shows an example. When $p_j = p_i$, the wedge does not stab $\mathcal{CH}(P)$ and the stabbing interval is $\{i\}$.

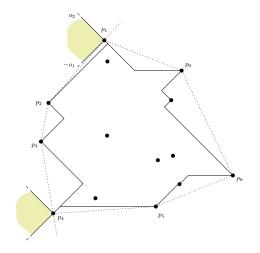
Figure 3: The stabbing interval of the stabbing wedge (a) in Figure 2 is $[1, 4]$.

Observation 1 *If j belongs to the stabbing interval of $w_{o_i o_{i+1}}$, then the orientation of the edge e_j of $\mathcal{CH}(P)$ belongs to the interval (o_i, o_{i+1}) in $\mathcal{O}$. Thus, when $\mathcal{O}$ contains the h supporting lines of the edges in E, the stabbing interval of all stabbing $\mathcal{O}$-wedges is a point and therefore $\mathcal{CH}(P) = \mathcal{OH}(P)$.*

Note that, as in the orthoconvex case, $\mathcal{OH}(P)$ can be disconnected. Actually, only opposite stabbing $\mathcal{O}$-wedges can overlap, generating a disconnected $\mathcal{O}$-hull: In any pair of non-opposite stabbing $\mathcal{O}$-wedges, one of them contains a bounding ray of the other one. As every wedge in the boundary of $\mathcal{OH}(P)$ is supported by at least two points in P, two non-opposite wedges that intersect would inevitably leave one of them non P-free, contradicting Proposition 1. Clearly, this is not the case with opposite wedges. See again Figure 2.

With this information we can compute the points of P which are θ_i-maxima, for $\theta_i = \pi - (o_{i+1} - o_i)$, in the orientation δ_i defined as the bisector of $w_{o_i o_{i+1}}$.

First, compute the stabbing interval for each pair of consecutive orientations in $\mathcal{O}$, and keep only those stabbing intervals which have at least 2 points of $\mathcal{CH}(P)$. In each of these different and consecutive stabbing intervals we will have different stabbing staircases. Each of these staircases is formed by stabbing wedges with aperture angle θ_i and oriented bisector δ_i. (Notice that, as in [4], we can also apply the technique by Avis et al. [2] selecting the points inside the arcs of circle with angle $\theta_i \geq \frac{\pi}{2}$ corresponding to the edges of $\mathcal{CH}(P)$ with the corresponding stabbing intervals and compute which of them are θ_i-maxima.) We compute their corresponding maxima points as the points whose angular intervals are pierced by the vertical lines δ_i in the table of orientations, so there will be at most $2k$ vertical lines piercing angular intervals.

So, we compute the θ_i-maxima points and their corresponding wedge given by two rays with aperture angle $\alpha \geq \theta_i$; this wedge is translated into an angular interval of angle α. Recall that since $\theta_i \geq \frac{\pi}{2}$, then, a point $p_i \in P$ has at most 3 disjoint angular intervals with angles denoted by α_i, β_i and γ_i. Thus, in the table of Figure 4 each interval has a pointer to its aperture angle value α_i, β_i or γ_i. Then, when a vertical line δ_j pierces an angular interval, to select this angular interval it has to be checked (in constant time) whether the aperture angle of the angular interval is bigger than or equal to the aperture angle corresponding to δ_i.
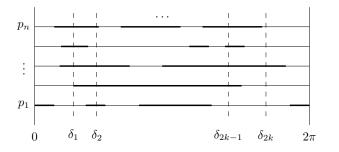

Figure 4: The table with the angular intervals and the $2k$ vertical lines δ_i.

Thus, we just have to arrange the angular intervals in a table as in Figure 4. Knowing the angular intervals corresponding to points of P pierced by the $2k$ vertical lines, we know which are the points at the steps of the staircases formed with the stabbing wedges w_i. Using standard techniques [10], we can compute these staircases and join them in $O(n \log n)$ time and $O(n)$ space to form the boundary of $\mathcal{OH}(P)$.

Once we have computed $\mathcal{OH}(P)$ in $O(n \log n)$ time and $O(n)$ space, it remains to show optimality of the time complexity: Given $\mathcal{OH}(P)$, we can compute in linear time $\mathcal{CH}(\mathcal{OH}(P)) = \mathcal{CH}(P)$, and it is known that computing the convex hull of a set of points in the plane has an $\Omega(n \log n)$ time [10]. We get:

Theorem 1 *The oriented $\mathcal{OH}(P)$ can be computed in $\Theta(n \log n)$ time and $O(n)$ space.*

Observation 2 *The complexities of the computation of the oriented $\mathcal{OH}(P)$ are independent of the number $2k$ of orientations.*

2.2 Computing the unoriented $\mathcal{OH}(P)$

Notice that $\mathcal{OH}(P)$ changes if we rotate the coordinate system, being orientation-dependent. Nevertheless, we can update the at most $2k$ staircases in $O(\log n)$ time per insertion or deletion of a point in some of the staircases. For this operation we need to maintain the (at most) $2k$ staircases into the (at most) $2k$ different information structures (balanced trees), one for each staircase. Notice that some of the staircases may appear and/or disappear during the rotation. The total insertion or deletion operation can be done in $O(kn(\log k + \log n)) = O(kn \log n)$ time.

We only have to sweep the table of angular intervals with the $2k$ vertical lines stopping at any event (insertion or deletion of a point in some staircase), since the number of angular intervals for a point is at most 3, but it can be appearing into any of the $2k$ staircases during the rotation. Therefore, there are at most kn events, and each of them has to be updated or deleted from the corresponding possible staircase where it can appear or disappear. We get the following result:

Theorem 2 *To compute and maintain the unoriented $\mathcal{OH}(P)$ during a complete rotation of the coordinate system can be done in $O(kn \log n)$ time and $O(kn)$ space. If the number k of orientations is a constant, the time and space complexities are $O(n \log n)$ and $O(n)$.*

Corollary 1 *To compute the orientation of the coordinate system such that the unoriented $\mathcal{OH}(P)$ has minimum number of steps or minimum number of staircases, can be done in $O(kn \log n)$ time and $O(kn)$ space.*

3 Maximizing the area of the unoriented $\mathcal{OH}(P)$

Using techniques similar to those in [1], we can compute the orientation of the coordinate system where $\mathcal{OH}(P)$ has minimum (or maximum) area, given the set of orientations $\mathcal{O}$ as above, and (this is relevant for the algorithmic result) the assumption that $\theta_i \geq \Theta = \frac{\pi}{2}$.

This condition allows us to compute the set of angular intervals of all the Θ-maxima points of P, and also, the essential properties that imply at most a linear (in n) number of intersections between arcs of circumferences of the different angles θ_i for all the possible steps of the staircases in a complete rotation of the coordinate system. We need this for the computation of the start and stop events of the overlaps between opposite stairs. The list of these events generate a set of orientation intervals, in which the set of vertices and overlaps in $\mathcal{OH}(P)$ remains unchanged.

The last ingredient needed to mimic the techniques in [1] is to adapt, using only basic trigonometry, the formulas from [3] in order to express the area of $\mathcal{OH}(P)$ in terms of the rotation angle θ, as area$(\mathcal{P}) - \sum_i$ area$(\triangle_i(\theta)) + \sum_j$ area$(\Box_j(\theta))$, where $\mathcal{P}$ is the simple polygon obtained joining counterclockwise consecutive vertices of $\mathcal{OH}(P)$ by joining consecutive maxima, the overlaps are parallelograms $\Box_j$, instead of rectangles, and the triangles $\triangle_i$ are no longer right triangles. See Figure 5.
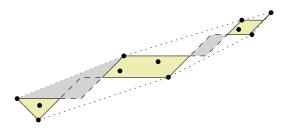

Figure 5: Expressing the area of $\mathcal{OH}(P)$: A triangle and a parallelogram are shaded.

As for the perimeter, the maintenance of the current perimeter is just given by the maintenance of all the current staircases, since the sum of the length of their steps gives the current perimeter.

Altogether, we obtain our final results:

Theorem 3 *To compute the orientation of the coordinate system such that the unoriented $\mathcal{OH}(P)$ has minimum (or maximum) area or perimeter can be done in $O(kn \log n)$ time and $O(kn)$ space.*

Corollary 2 *To compute the orientation of the coordinate system such that the unoriented $\mathcal{OH}(P)$ is connected or it has the minimum (or maximum) number of connected components, can be done in $O(kn \log n)$ time and $O(kn)$ space.*

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