Representing Orders on the Plane
by Translating Convex Figures

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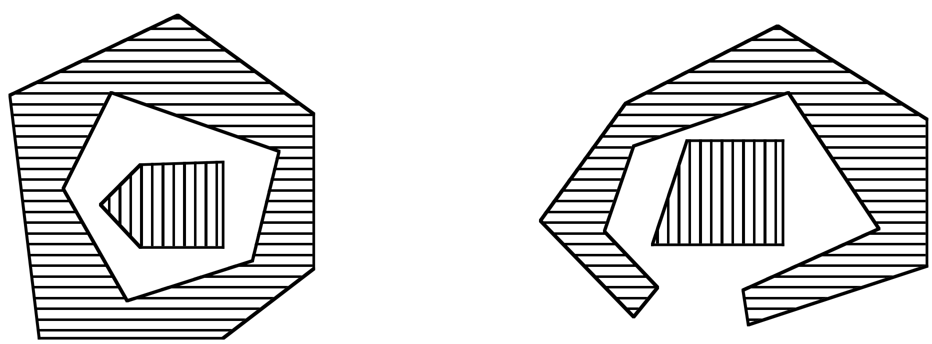
Introduction

How may a robot arm be moved to pick up a particular object from a crowded shelf without unwanted collisions? How may a cluster of figures on a computer screen be shifted about to clear the screen without altering their integrity and without collisions? These are instances of the problem known in computational geometry as the "separability problem". It is part of the recent and rapidly growing theme of "motion planning".

For our purposes we cast the problem as follows. Given a finite collection of disjoint figures in the plane, is it possible to assign to each a single direction of motion so that this collection of figures may be separated, through an arbitrarily large distance, by translating each figure one at a time, along its assigned direction? Of course, the figures themselves must satisfy some topological constraints, or else the answer is surely no. A disk located in the interior of an annulus cannot be separated from it by translations of any kind (see Figure 1a). Indeed, even figures topologically contractible to a point may not be separable without deforming them (see Figure 1b).

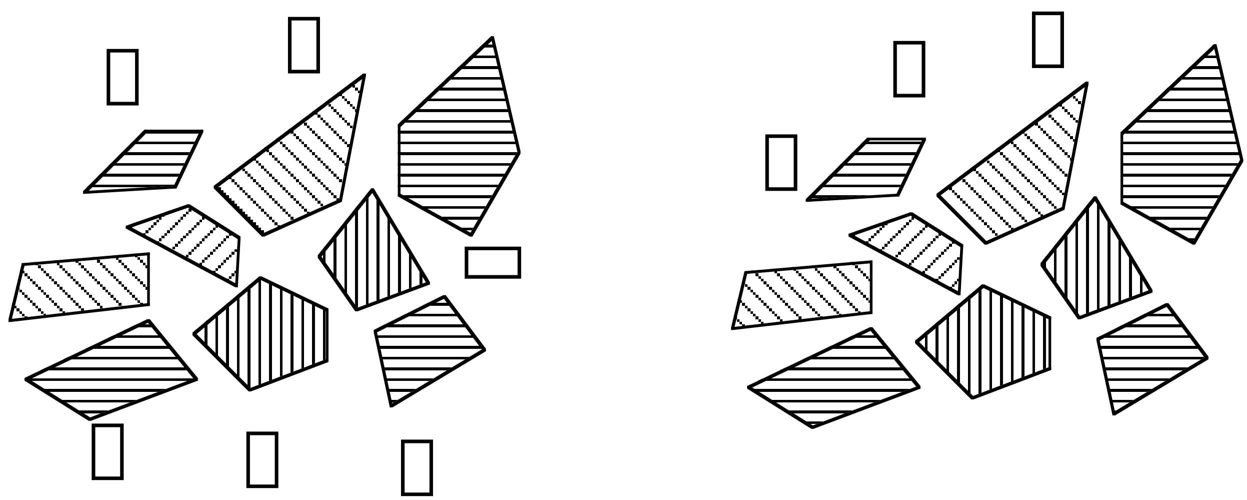


(a) (b)

Figure 1.

We shall consider only convex figures in the plane (see Figure 2a). Indeed, given a collection of disjoint, convex figures, the separability problem always has a positive solution. Loosely speaking, at least one of the convex figures is on the "outside" or "boundary" of the collection, and therefore it may be removed.

More generally, for any fixed direction of motion, at least one of the convex figures may be removed—a figure on the outside. Therefore by induction, figures may be removed, one at a time, along this axis of motion (see Figure 2b).



(a) (b)

Separable figures

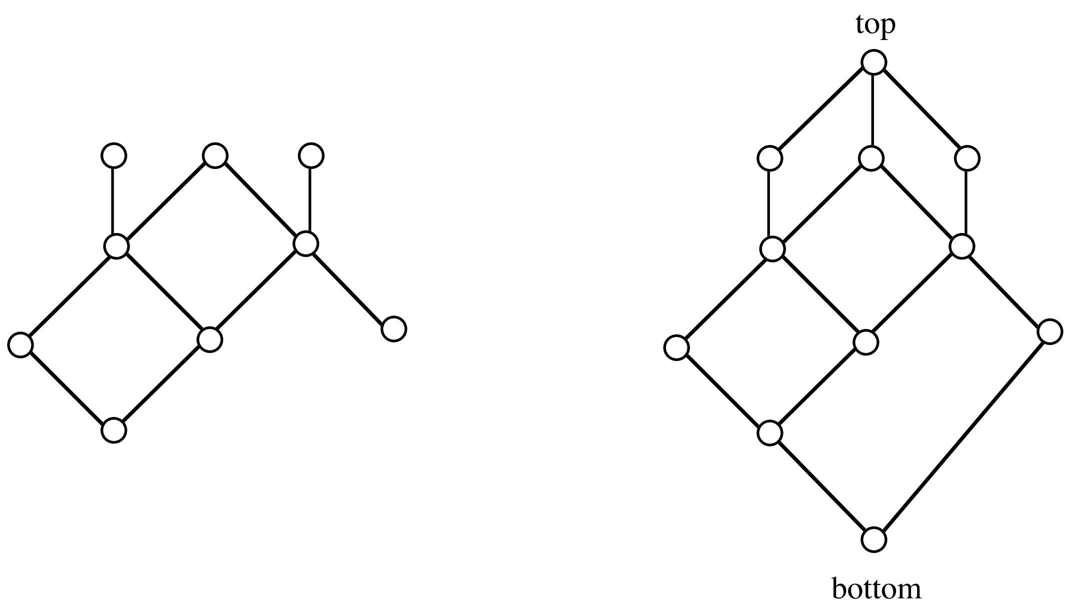
Figure 2.

We present a computational model for this separability problem based on the theory of ordered sets. Suppose each figure in the collection of disjoint convex figures is assigned a single direction of motion, not necessarily all the same. For figures A and B we say that B *obstructs* A if there is a line joining a point of A to a point of B which follows the direction assigned to A. We write A~B. More generally, we write A<B if there is a sequence $A = A_1 \sim A_2 \sim \ldots \sim A_k = B$. This relation $<$ is transitive. It is appropriate to call this binary relation $<$ a *blocking* relation. If the blocking relation has no directed cycles then it is antisymmetric too. In that case the blocking relation $<$ is a (strict) order on the set of these figures. If each of the figures is assigned the same direction, we call the relation *one-directional.* In that case, any maximal figure (with respect to <) is on the "outside".

Here is the first indication that this order-theoretical model is a striking one. It is our first result and our initial motivation.

Theorem 1. *There is a one-to-one correspondence between the class of all one-directional blocking relations and the class of all truncated planar lattices.*

A *lattice* is an ordered set in which every pair of elements has least upper bound and greatest lower bound; it is *planar* if its "diagram" can be drawn without any crossing lines. If the top and bottom elements of a planar lattice are removed it becomes a *truncated* lattice. See Figure 3a which corresponds to the collection of figures illustrated in Figure 2b.

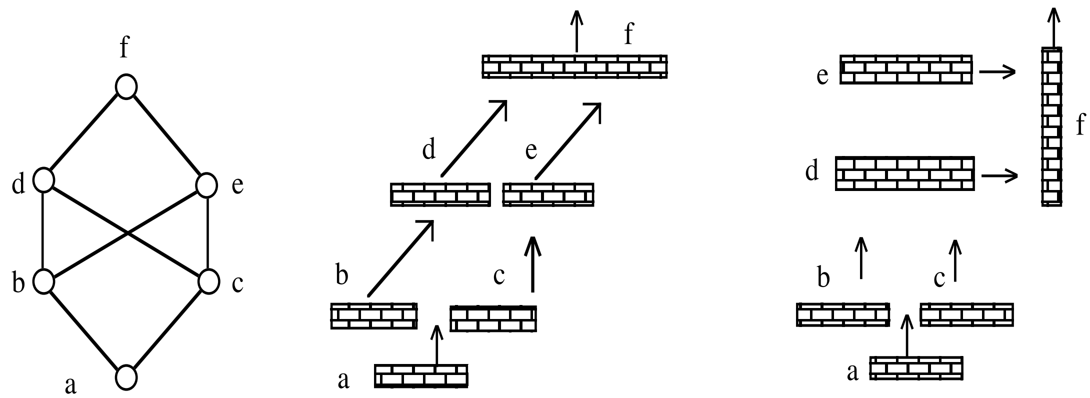


(a) A truncated planar lattice

(b) A planar lattice

Figure 3.

Ordered sets may come to play an important role as data structures for "motion planning" problems. We are naturally led to the question of describing the orders that correspond to collections of disjoint convex figures each assigned a direction of motion, not necessarily all the same. Indeed, given an n-element ordered set what is the minimum number of directions needed for a collection of n disjoint, convex figures, in order that its blocking relation coincides precisely with this initial ordered set? According to Theorem 1, a nonplanar ordered set cannot correspond to a one-directional blocking relation (see Figure 4).



A nonplanar ordered set P

A two-directional representation of P

Another two-directional representation of P

Figure 4

We say that a collection of disjoint convex figures, each assigned one of m directions, is an m-*directional representation* of an ordered set P, if its blocking relation is identical to the ordering of P. Thus, given P, what is the least integer m such that P has an m-directional representation? Does every ordered set even have an m-directional representation, for some m?

In an important sense the answer to the first question is m=2.

Theorem 2. *Every ordered set has a subdivision which, in turn, has a two-directional representation.*

This contrasts sharply with the striking answer to the second question.

Theorem 3. *There is an ordered set which has no m-directional representation at all, for any positive integer m.*

The separability problem has received extensive attention in recent years. A typical result is that the order of separation of n disjoint rectangles in the plane in a given direction can be determined in O(n log n) time [L. J. Giubas and F. F. Yao (1980)]. Others have considered this very same issue for other types of figures (e.g. star-shaped) (cf. [B. Chazelle, T. Ottmann, E. Soisalon-Soinen and D. Wood (1983)], [M. Mansouri and G. T. Toussaint (1985)], [G. T. Toussaint (1985)]). [J.-R. Sack and G. T. Toussaint (1985)], for instance, showed that determining all directions of separability for a given pair of n-polygons can be done in O(n log n) time.

It is a remarkable fact, rediscovered from time to time that, in three dimensions, there are collections of convex bodies such that none can be moved without disturbing others (cf. R. Dawson (1984)).

In an entirely different setting, G. X. Viennot (1985) has considered a combinatorial notion which he has called a "heap of pieces". It may be thought of as a collection of "lego-like" blocks (the same ones beloved by our children), put together in some fashion and ordered by, $A<B$ if the removal of A from the pile involves the previous removal of B. This seems to be a valuable computational model of enumeration for it has far-reaching consequences, for example in statistical mechanics.

Another question that may be related is raised by B. Sands (1985). How many different slopes are needed in the diagram representation of an ordered set? Sands has conjectured that for lattices the maximum of the "up-degrees" and the "down-degrees" will do. For distributive lattices this is true. Diagrams, however, are unlike blocking relations in that a given circle representing a vertex may be assigned several different directions, one for each upper cover.

One-directional Representations and Planar Lattices

What do blocking relations have to do with planar lattices?

To answer this question we need to review some basic facts about ordered sets and their pictorial representations in the plane. Let P be an ordered set. For elements a and b in P, we say that b *covers* a or a is *covered* by b, in symbols $b \succ a$ or $a \prec b$, if $b > a$ and, for each c in P,
$b > c \geq a$ implies $c = a$. We also call b an *upper cover* of a or a a *lower cover* of b. The graph $\mathrm{cov}(P)$, whose vertices are the elements of P and with an edge joining a pair a, b of vertices whenever either $a \prec b$ or $b \succ a$, is called the *covering graph* of P. If there is a blocking relation that corresponds to the ordered set P, then the upper covers of a convex figure A are those convex figures obstructing A and minimal in the blocking relation with respect to this property. Such upper covers B of A we shall also render with the symbol B>-A or A-<B.

While important in the study of ordered sets, the covering graph of P is only one aspect of the pictorial scheme that is in common use to illustrate ordered sets. It is the custom to represent P pictorially on the plane by means of its *diagram* in which small circles, corresponding to the elements of P, are arranged in such a way that, for a and b in P, the circle corresponding to b is higher than the circle corresponding to a

whenever $b > a$ and a straight line segment connects the two circles whenever b covers a.

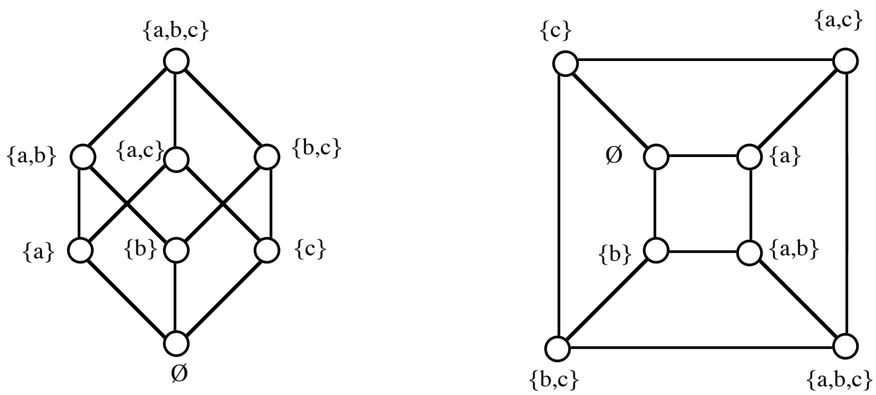
Obviously the covering graph and the diagram of P are closely related. We may assign arrows to the edges of cov(P) according to the rule a∅b if a –< b. Then, exploiting the antisymmetry of the order relation we may orient the *directed* covering graph in such a way that all arrows make an angle with the horizontal satisfying:

$$0^\circ < \square < 180^\circ.$$

In other words, position the vertices so that all arrows point "upwards". This done, erase all arrows, et voilà, the diagram!

Nevertheless, the closeness of the relationship between cov(P) and the diagram of P is over-rated. Take the concept of planarity. We say that (the ordered set) P is *planar* if it has a diagram in which none of the lines corresponding to the covering pairs intersect, except possibly at an endpoint, where they meet a small circle corresponding to an element of P. Such a rendering of P we call a *planar representation.*

An ordered set may be nonplanar, yet its covering graph (as a graph) may be planar in the usual sense for graphs, that is there is a representation of cov(P) with no edges crossing. An example is the ordered set $\mathbf{2}^3$ of all subsets of a three-element set {a,b,c} ordered with respect to set inclusion. Its diagram has no planar representation at all, thus it is a nonplanar ordered set. The covering graph of $\mathbf{2}^3$, however, is a planar graph (see Figure 5).



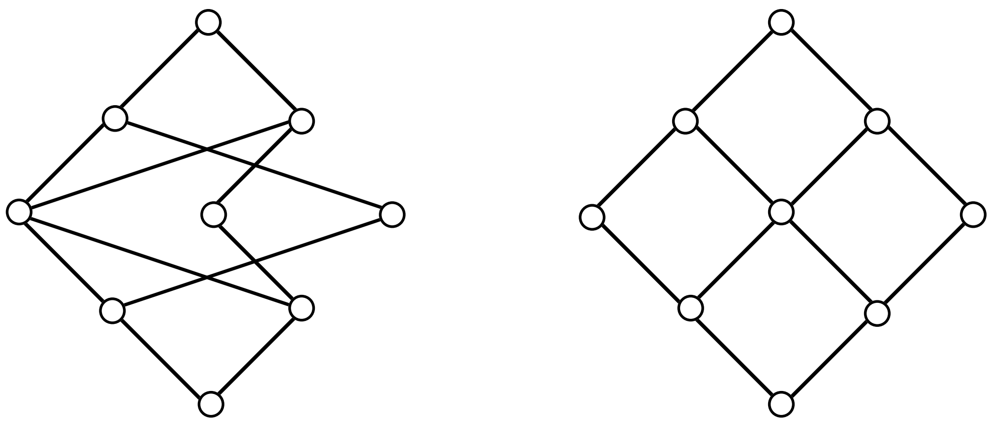
A diagram of $\mathbf{2}^3$
It has no planar representation.

The covering graph of $\mathbf{2}^3$ is a planar graph.

Figure 5.

A planar ordered set may, of course, have a nonplanar representation (see Figure 6).

The edges of graphs are often rendered as arcs—not necessarily straight line segments. It is a well-known result [K. Wagner (1936)](cf.[I. Fáry (1948)]) that every planar graph has a planar representation in which each edge is drawn as a straight line segment. There is an analogous result for the diagram of an ordered set which is, however, drawn in terms of "monotonic" arcs.



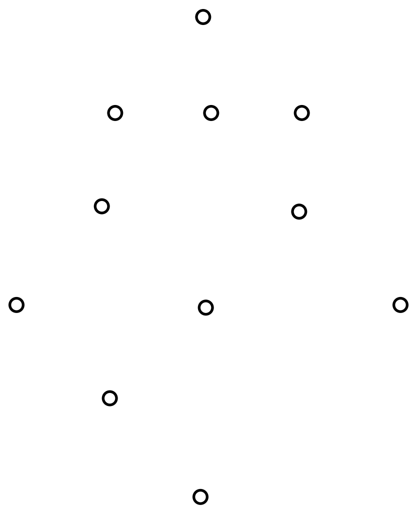
A non planar representation of an ordered set.

A planar representation of the same ordred set

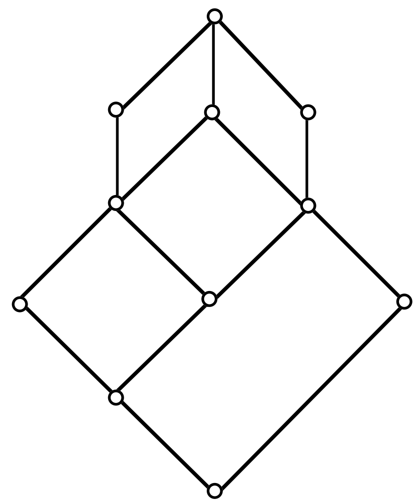
Figure 6.

Call an arc (in the plane) *monotonic* if no two distinct points on it have the same y-coordinate. It is convenient to relax the convention that the covering relation in the diagram of an ordered set be drawn using just straight line segments. Indeed, we may allow these edges to be drawn as monotonic arcs, for the point is that, according to a result of D. Kelly (1987), if an ordered set has a diagram using monotonic arcs which is planar then it also has a diagram using straight line segments which is planar.

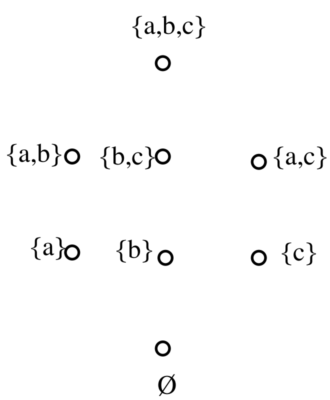
Planar lattices occupy an unexpectedly important place in the theory of ordered sets. For one thing, a finite lattice is planar if and only if it has dimension at most two. [The *dimension* of an ordered set is the least number of linear extensions whose intersection coincides with the ordered set itself.] Moreover, a finite ordered set has dimension at most two if and only if its normal completion is a planar lattice (cf.[D. Kelly and I. Rival (1975)]. [The *normal completion* of a finite ordered set P is the smallest lattice that contains it as a subset.] While planar lattices, on the one hand, and ordered sets of dimension two, on the other hand, are thus intimately related, there are important distinctions to draw. For instance, for any positive integer n, there is a finite planar ordered set which has dimension n (cf. [D. Kelly(1981)]. Even planar ordered sets of dimension two may not have a one-directional blocking relation. Planar lattices do have a one-directional blocking relation.



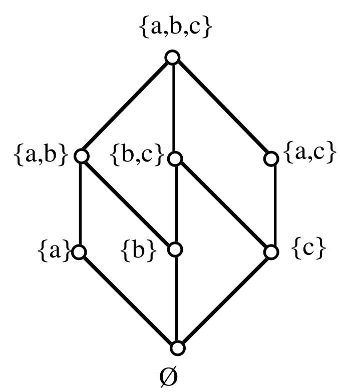
A planar representation (using monotonic arcs) of a planar lattice.



A planar representation (using straight line segments) of this planar lattice.



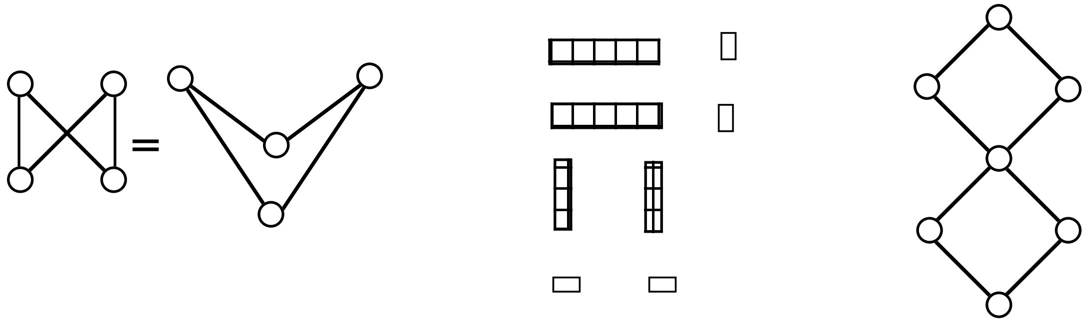
A diagram of $\mathbf{2}^3$ using monotonic arcs, $\mathbf{2}^3$ is a non planar ordered set.



An improper diagram of $\mathbf{2}^3$. The arc from {a} to {a,c} is not monotonic. $\mathbf{2}^3$ has a planar covering graph.

Figure 7.

In fact, if all ot the convex bodies are assigned the same direction then it is convenient to visualize the obstruction and blocking relations in a more particular manner. Actually A-<B if and only if there is a segment joining a point in A to a point in B following the direction assigned to A and, any such segment line passes through no other figure. In this one directional case, we also say that B is *visible* from A.



A planar ordered set of dimension 2.

It has a two-directional collision relation, but no one-directional collision relation.

The normal completion is a planar lattice. It has a one directional collison relation.

Figure 8.

Theorem 1. *There is a one-to-one correspondence between the class of all one-directional blocking relations and the class of all truncated planar lattices.*

Proof. *Part I.* We shall first show that, to any one-directional blocking relation, we may assign (in a one-to-one fashion) a truncated planar lattice.

Suppose we are presented with a collection of disjoint convex figures on the plane, each assigned the same direction of motion, parallel to the y-axis, say. As we have already observed, such a collection is always separable. Moreover, it is easy to verify that any sequence of translations of the figures without blocking, along the (one) assigned direction of motion, leaves the blocking relation unchanged. Formally speaking, we might say that the blocking relation is an "invariant" of the direction of motion. It is simple to check that this feature is no longer true for $m \geq 2$ directions. Thus we may suppose that no two of the figures ever occupy a common y-coordinate; in fact, we may suppose that there is some □>0 such that about each figure there is a vertical □-band free of any other figure. We may suppose too that □-bands of adjacent figures are disjoint too. (See Figure 9). For figures A, B, if A–<B, that is, if B is visible from A, then there is a vertical ray from A to B. Place a small circle somewhere inside each figure. (If the figure is an interval, chose a small circle (of radius much less than □)

incident with it.) Now, if A–<B , join the circle representing A to the intersection of the AB-ray and the □-band of A, and join the intersection of the AB-ray and the □-band of B to the circle representing B. These three segments from A to B constitute a monotonic arc from A to B.

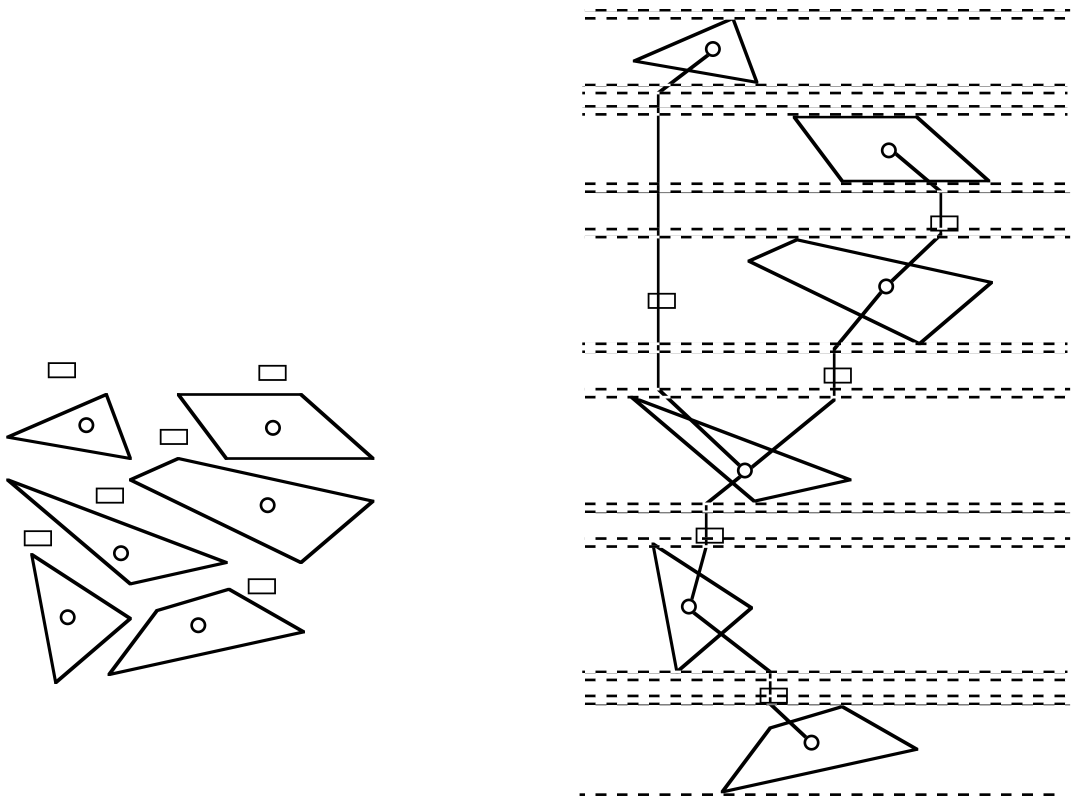


Figure 9.

We claim that this collection of small circles and monotonic arcs is a planar representation of a truncated lattice.

First we verify that it is a planar representation. For contradictions, suppose that two monotonic arcs intersect, the arc from A to B and the arc from C to D. Each of these arcs consists of three line segments $\square_1$, $\square_2$, $\square_3$ where $\square_1$ joins the circle a for A with the vertical ray corresponding to the visibility ray $\square_2$ from A to B and $\square_3$ joins $\square_2$ to the circle b for B. Similarly $\square_1$, $\square_2$, $\square_3$ is the corresponding triple of line segments from the circle c for C to the circle d for D. There are, in all, nine cases to consider: $\square_i$ crosses $\square_j$, i,j=1,2,3. If $\square_2$ ever coincides with $\square_2$ then either A is above C or C is above A; either case is impossible for visibility rays cannot be broken. Thus we may suppose that $\square_2$ and $\square_2$ are disjoint, say $\square_2$ has smaller x-coordinate than $\square_2$.

Now, if $\square_1$ intersects $\square_1$, say, then again A and C are comparable. The other seven cases all come down to this same contradiction, too.

Next, we must verify that it is a truncated lattice. In effect, every pair a, b of elements which has an upper bound at all must have least upper bound. Suppose that c and d are both minimal upper bounds of a and b. In the planar representation there are sequences of monotonic arcs from a to c, a to d, b to c and b to d. We may suppose that the y-coordinate of c is larger than the y-coordinate of d. By planarity d is located inside the region bordered by the monotonic arcs from a to c, b to c and a line segment joining a to b (see Figure 10). Then the (vertical) visibility ray from the figure D corresponding to d will surely pass through the path of monotonic arcs from a to c or b to c. (Note that d may be a maximal element of the order relation, in which case there are no arcs actually emerging from it in the diagram.) And, in any case, there will be a first element e such that $a \leq e \leq c$ or $b \leq e \leq c$ whose corresponding figure is visible from D. Then, according to the transitivity of the blocking relation, $d<c$, although both c and d were supposed to be minimal upper bounds of a and b.

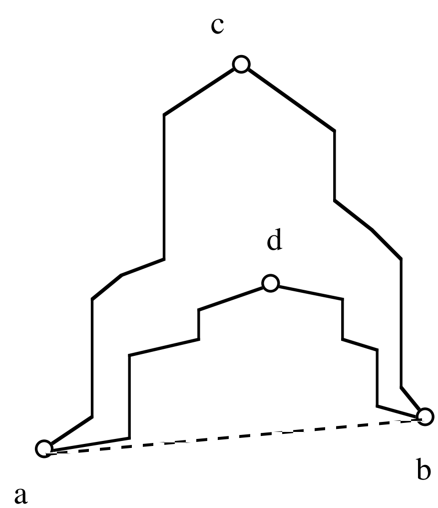


Figure 10.

As the (directed) covering relation of the truncated planar lattice corresponds to the visibility relation of the original blocking relation, the mapping constructed here from the class of all blocking relations to the class of all truncated planar lattices is one-to-one.

Part II. It remains to verify that there is a one-to-one mapping from the class of all truncated planar lattices to the class of all one-directional blocking relations. Let L be a truncated planar lattice. Let $L' = 0 \oplus L \oplus 1$ be the planar lattice corresponding to L with bottom element 0 and top element 1, both adjoined, and suppose that a planar representation (with straight line segments) of L' is given. This, of course, induces a planar representation of L. We shall require some further facts about planar lattices and their representations. For details see [D. Kelly and I. Rival (1975)]. The salient facts are these. A horizontal line which intersects this representation has an intersection point with least x-coordinate. The collection of all these intersection points constitutes the *left boundary* of L', in effect, a maximal chain (from 0 to 1). The *left side* of a maximal chain C of L' is the region of this representation defined by the left boundary of L' and C. For each $a \in L'$ the planar representation induces a strict linear ordering on the set of its upper covers. Let b and c be distinct upper covers of a and suppose that the y-coordinate of b is less than the y-coordinate of c. Let b_1 be the x-coordinate of b and let c_1 be the point of intersection of the line segment joining a to c with the horizontal through b. Then b is *to the left of* c if $b_1 < c_1$ and, similarly, c is *to the left of* b. (See Figure 11). For noncomparable elements u, v of L' we define the relation $u \lhd v$ if there are upper covers u', v' of the infimum of u and v such that $u' \leq u$, $v' \leq v$ and, with respect to the given planar representation, u' is to the left of v'. (In effect, u is left of v.)

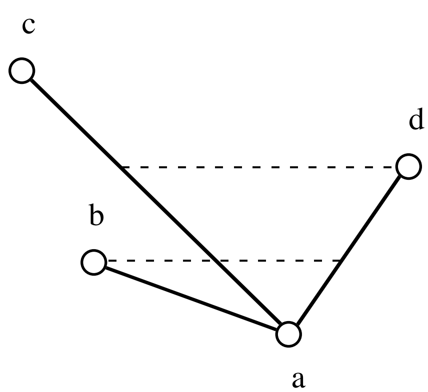


Figure 11.

Here are two simple yet important observations: if $u \lhd v$ then u is on the left side of any maximal chain through v; if u is noncomparable to v and u is on the left of some maximal chain through v then $u \lhd v$. A final preliminary fact is that there is always an element a in L on the left boundary of a planar representation such that a has precisely one lower cover and precisely one upper cover. Such an element is called *doubly irreducible*, for it is both infimum irreducible and supremum irreducible.

We proceed by induction on the cardinality of L to show that, for a given planar representation of L (as induced by L'), there is a collection of disjoint convex figures, actually, all horizontal line segments, and all assigned the same upward vertical direction, whose blocking relation is precisely the order of L and whose arrangement on the plane corresponds (according to the construction of Part I) to the given planar representation of L. To begin, let a∈L be a doubly irreducible element on the left boundary of the planar representation of L'. Let K = L - {a} . Then there is a planar representation of K induced by L. By the induction hypothesis we may suppose that there is a collection of disjoint, horizontal line segments on the plane which, if each is assigned the same upward vertical direction, has blocking relation which corresponds to K and, whose arrangement on the plane corresonds to the given planar representation of K.

The final step is to construct a horizontal line segment for a. Let b be the unique upper cover of a in L' and let c the unique lower cover of a in L. Both b and c lie on the left boundary of the planar representation of L'. If b=1 and c=0 then inserting a horizontal line segment whose largest x-coordinate is less than the x-coordinates of all other line segments will do. If b<1 and c=0 then we may extend to the left the left endpoint of the segment B corresponding to b and construct a horizontal line segment for a just below this extension. The case b=1 and c>0 is similar. Thus, we may suppose that 0<c–<a–<b<1.

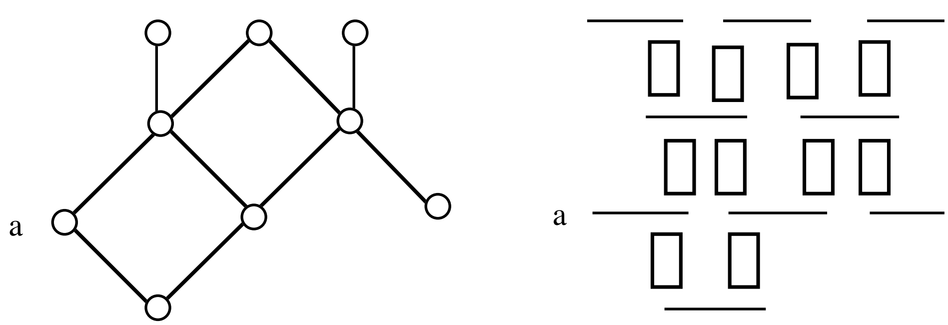


Figure 12.

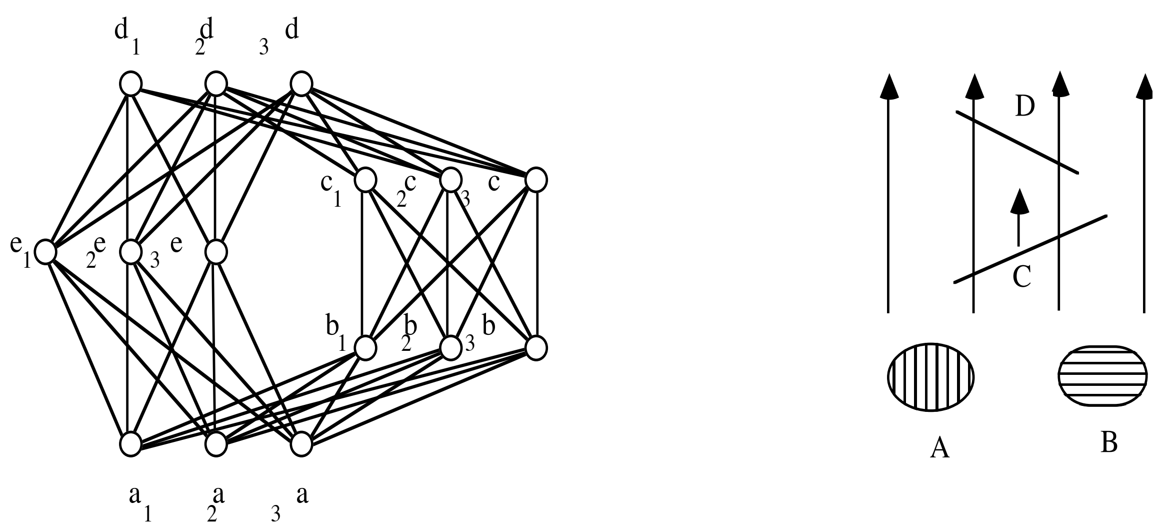
If in the arrangement of horizontal line segments corresponding to K, there is a vertical line segment joining a point of C corresponding to c to a point of B without passing through any other figure,then we may simply insert a horizontal line segment A just large enough to break this line segment from C to B. Thus, let us suppose that no such vertical line segment exists from C to B. In this case we propose to extend both B and C to the left until such a vertical line segment from C to B may be constructed. As long as this can be done we may insert A as above. The only obstacle that may arise is that, with the extension of C, say, another line segment D obstructsC although c is noncomparable (in L) with the element d corresponding to D. In this case, however, d □ c although c lies on the left boundary of L. This is a contradiction. Thus, C may be extended and, similarly, D may be extended too.

Once constructed, it follows that the planar representation corresponding to this arrangement has the same □□relation as the original planar representation of L (as induced by L'). The correspondence is, therefore, one-to-one.

Two-Directional Representations and Subdivisions

It is natural to ask for a "higher dimensional" analogue of Theorem 1. At this time we do not know which ordered sets have a two-directional representation. Judging from Theorem 1, this would seem to be an interesting direction of investigation. We have shown elsewhere that every interval order has a two-directional representation (see R. Nowakowski, I. Rival and J. Urrutia).

In the proof of Theorem 1 we have seen that every truncated planar lattice has a one-directional blocking relation using only line segments. In fact, it is fairly obvious that, for the minimal elements, we may use only line segments. Indeed, for any convex figure corresponding to a minimal element we may replace it by any diameter which touches both extremities of the "light beam" it produces.



(a) (b)

Figure 13.

On the other hand, there are ordered sets of dimension two which require at least three directions in any blocking representation using disjoint convex figures. For instance, the ordered set illustrated in Figure 13a (which has dimension two—it is even series-parallel) requires at least three directions. Here is the reason. First, the four-cycle $\{a<c, a<d, b<c, b<d\}$ requires at least two directions (as we know from Theorem 1). A simple and direct reason in this special case is that if the disjoint convex figures A and B, corresponding to a and b, were both assigned the same direction, say upward vertical, then both of the disjoint convex figures C and D, corresponding to c and d, must obstruct from both A and B. According to the convexity of C, there is a line entirely in C which is met by vertical rays from both A and B; similarly for D (see Figure 13b). Then if C and D also follow the same direction , one will necessarily obstruct the other.

Now, if the ordered set illustrated in Figure 13a has a two-directional blocking relation, then, in any case, two of its three minimal elements a_1, a_2, a_3 must have the same direction. Say a_1, a_2 both have the same direction $\square_1$. Then, according to the observation above at least two of b_1, b_2, b_3 must have a different direction , say $\square_2$. Furthermore, we may suppose that both c_1, c_2 have direction $\square_1$ and d_1, d_2 the direction $\square_2$. On the other hand, for the same reasons, we may suppose that e_1, e_2 both have the direction $\square_2$. Then the four-cycle $\{d_1< e_1, d_1< e_2, d_2< e_1, d_2< e_2\}$ apparently has a one-directional blocking relation, which is a contradiction.

The point of this ordered set is that it is the "lexicographic sum" of the "pentagon" $\{a<b<c<d, a<e<d\}$ (as index set) with three-element antichains $\{x_1, x_2, x_3\}$ (as blocks). Of course, any odd polygon as index set with large antichain blocks will do.

Nevertheless, with respect to subdivisions, two directions suffice. A *subdivision* of an ordered set P is an ordered set Q obtained from P by adjoining to a subset of covering pairs $\{a_i{-\!\!<}b_i \mid i\square\mathbf{I}\}$ a set of new elements $\{c_i \mid i\square\mathbf{I}\}$ with comparabilities $a_i{-\!\!<}c_i{-\!\!<}b_i$ and those induced by a_i and b_i, $i\square\mathbf{I}$.

Theorem 2. *Every ordered set has a subdivision with a two-directional representation.*

In effect, the idea behind the proof is that a figure may block another without obstructing it. Thus, $A<B$ with respect to the blocking relation, yet A need not be covered by B, that is, B need not obstruct A. Indeed, it may even be that no ray, starting from A in the direction assigned to A, will strike B even if extended arbitrarily (see Figure 14).

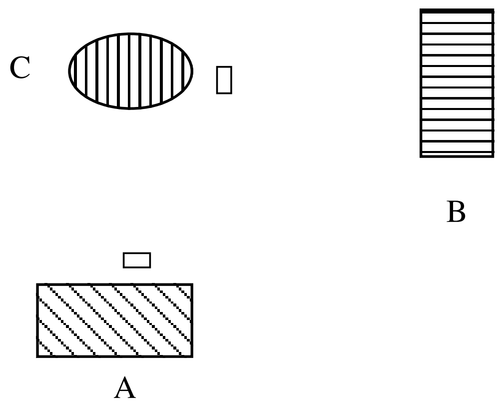


Figure 14.

Proof. Let P be an ordered set and let L be any linear extension of P, that is, a total order on the same set as P such that a<b in P implies a<b in L. Assign to each element of P a vertical line segment of unit length with midpoint placed along the y=x line on the plane at two unit intervals apart and according to the total order of the linear extension. Notice that no two line segments have a common y-coordinate. To each such segment we assign the upward vertical direction. The corresponding blocking relation is evidently an antichain—no segment obstructs any other!

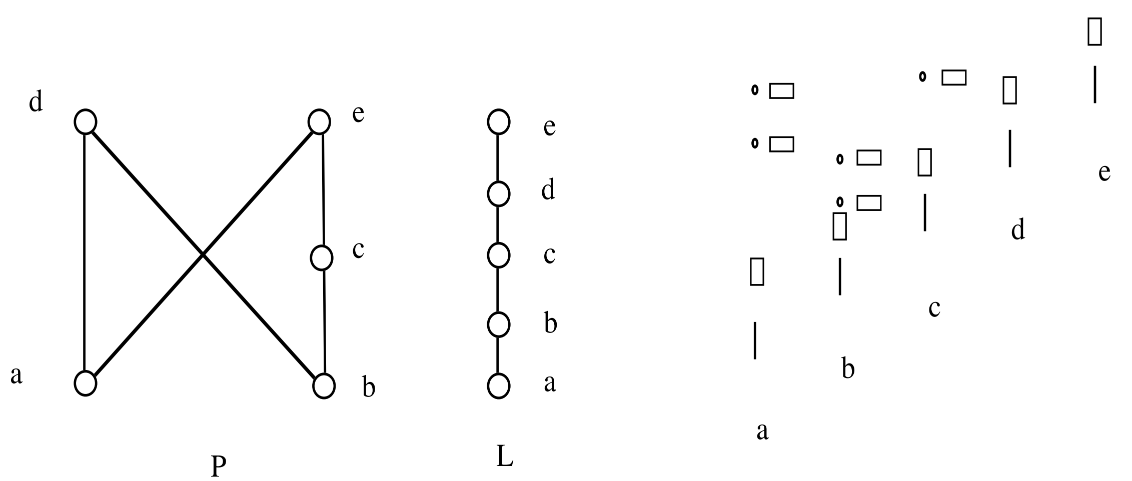


Figure 15

A simple artifice restores the order. For any line segment B corresponding to b in P and each of its lower covers $A_1, A_2, \ldots$ corresponding to elements $a_1, a_2, \ldots$ in P, we adjoin points $C_1, C_2, \ldots$ each with distinct y-coordinate between the least and greatest y-coordinate of B and with x-coordinate identical to $A_1, A_2, \ldots$, respectively. Each such point is assigned the positive horizontal direction. Then B obstructs each C_i

and C_i, in turn, obstructs A_i. In effect, we have constructed a collection of disjoint convex figures corresponding to that subdivision of P in which, for each covering pair a–<b, there is introduced a new element a–<c–<b. (See Figure 15.)

Many-Directional Representations and Non-Representable Ordered Sets

Before we proceed to the proof of Theorem 3, we examine several examples related to it.

Two of the examples are intended to illustrate the claim (weaker than Theorem 3) that, *for any positive integer m, there is an ordered set with no m-directional representation.*

The first example is a generalization of the ordered set illustrated in Figure 13a. We begin with an ordered set whose covering graph has chromatic number m+1. (That such ordered sets exist is, in the first instance, due to P. Erdös (1959).) Now, for each element, substitute an (m+1)-element antichain, all of whose elements inherit just the comparabilities of the elements of the underlying ordered set. This ordered set is the lexicographic sum of the m+1-chromatic ordered set (as index set) with (m+1)-element antichains (as blocks). If this lexicographic sum has an m-directional representation,then as each block has m+1 elements, at least two of its elements must be assigned the same direction. Following the same argument as we used for the ordered set in Figure 13a, any two elements which cover these two cannot both be assigned this same direction too. It follows that if B_1, B_2 are blocks of this ordered set such that each element of B_2 covers each element of B_1, then there is a pair of elements in B_1 assigned the same direction $\square_1$ and a pair of elements of B_2 assigned the same direction $\square_2$ and $\square_1 \neq \square_2$. In effect, these direction assignments induce a colouring of the covering graph of the index set of this lexicographic sum which, however, is a contradiction, for it has chromatic number m+1.

Let C(n,m) stand for the binomial coefficient of the number of m-element subsets of an n-element set.

The second example is probably much smaller. It is a bipartite ordered set with 3m+1 minimal elements and C(3m+1,2) maximal elements, one for each distinct pair of minimals; that is, for each distinct pair of minimals there is precisely one common upper bound. Suppose this ordered set has an m-directional representation. Then four of its minimal elements must correspond to convex figures each assigned the same direction. We may suppose that the figures A, B, C, D corresponding to the minimal elements a, b, c, d are each assigned the same upward vertical direction. Therefore, they can have no common x-coordinates. We may suppose that the x-coordinates of A are less than those of B, B less than C, and C less than D. Now, there is a convex figure A+C=supremum{A,C} which corresponds to a+c=supremum{a,c} . Because A+C is convex, there is a line in A+C with an x-coordinate common to A and one common to C. If B has y-coordinates less than this line, then B<A+C (for B, too, is assigned the upward vertical direction) although b<a+c. Thus, B has y-coordinates larger than those of the line constructed from A+C. We consider B+D. It should not "obstruct" C for c<b+d. B+D contains a line with an x-coordinate common to B and one common to D. Then D must have y-coordinates smaller than those of C. In particular, this line of B+D must pass "below" C; it will, therefore, intersect the line constructed for A+C. Thus A+C and B+D are not disjoint after all (see Figure 16).

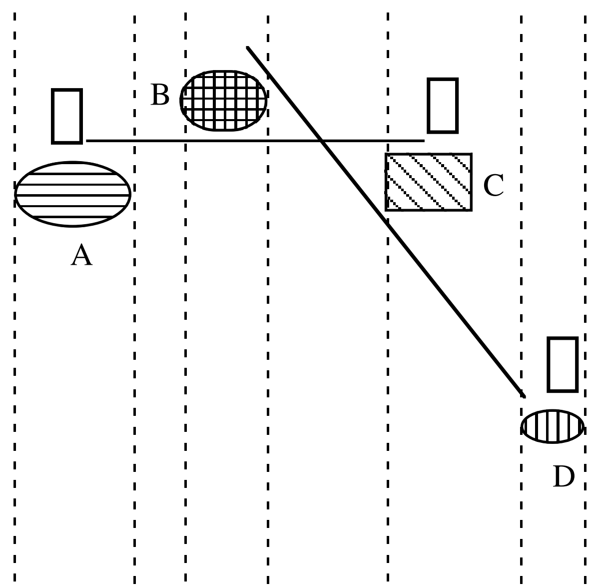


Figure 16

In a sense, this argument is implicit in the analysis of the four-cycle and its representation (see Figure 13b).

The third example is of an ordered set which has no m-directional representation at all, provided that is, that its maximal elements are represented by points alone. This example, too, is bipartite. It has six maximal elements and C(6,3) minimals, each a lower bound to a distinct triple of the maximals. Suppose that this ordered set has a blocking representation. The minimals in it may, as always, be represented by line segments. Suppose that there is a representation in which the six maximals are represented by six points $P_1, P_2, P_3, P_4, P_5, P_6$. As each triple of maximal elements of the blocking relation is visible from a distinct minimal element, it follows that, for any three distinct points P_i, P_j, P_k, the triangle they generate contains no other point. This, in turn, implies that these six points form the sides of a convex hexagon. We may suppose that, after possibly relabelling, the points $P_1, P_2, P_3, P_4, P_5, P_6$ are consecutive vertices of this hexagon.

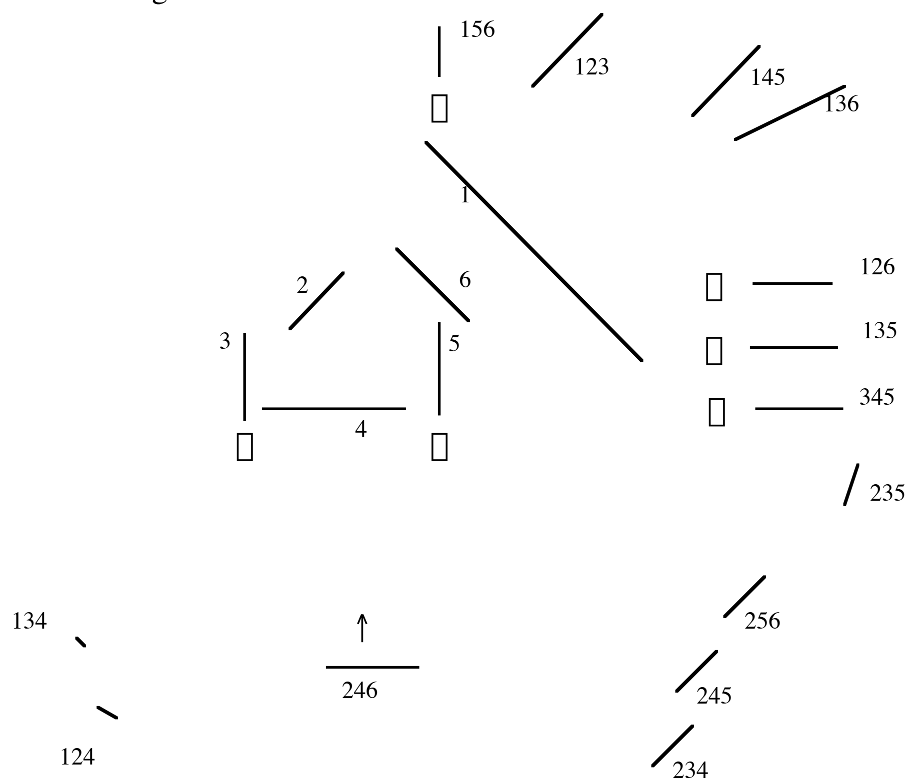


Figure 17

Finally, the line segments L_{135} and L_{246} which obstruct the points P_1, P_3, P_5 and P_2, P_4, P_6, respectively, will either intersect or give rise to unwanted obstructions, either of which is impossible.

On the other hand, if line segments are used for the six maximal elements, this ordered set does have a blocking representation (see Figure 17).

Theorem 3. *There is an ordered set which has no m-directional representation, for any positive integer m.*

Proof. Let P_n be the ordered set whose underlying set consists of all nonempty subsets of $\{1, 2,\ldots,n\}$ ordered by $a \leq b$ just if, there is some $i=1, 2,\ldots,n$, such that $a = \{i\}$ and b is a subset containing i. In effect, P_n is the ordered set in which the only comparabilities are, according to set inclusion, between the singleton and any other subset. Any two subsets, neither singletons, are noncomparable in P_n. Then P_n is an ordered set in which every chain has at most two elements. We shall show that, for any n large enough, ($n \geq 50$ will do), P_n has no m-directional representation, for any positive integer m.

Suppose, on the contrary, that, for some large n, and some m, P_n does have an m-directional representation. As P_n is bipartite, the blocking relation coincides precisely with the visibility relation: for every subset of $\{1, 2,\ldots,n\}$ there is a convex figure which is visible just from the convex figures representing the singletons corresponding to its members. We may suppose that the convex figures assigned to the singletons, as minimals, are line segments. In effect, we may visualize the minimals as a set of fixed searchlights, each producing a beam of light. The beam width is determined by the line segment representing the corresponding minimal element and the direction assigned it. The maximals, in turn, must be a set of disjoint convex figures intersecting corresponding light beams. Distinct maximals must intersect distinct subsets of these beams.

In fact, we may suppose that any convex figure corresponding to a maximal element intersects at least one of the bounding lines of the beam it obstructs (for if it does not, we may extend it until it does).

It is convenient, and no loss in generality, therefore to think of the n beams as $k=2n$ straight lines on the plane. A simple recurrence relation shows that there are at most $1 + k(k+1)/2$ regions or faces on the plane. The maximum is attained in the case that every pair of lines intersects, and no three lines intersect at a common point. Each of these faces is convex, for each is the intersection of half-planes. Moreover, 2k of these faces are unbounded.

We shall show that there can be no more than $o(n^8)$ such convex figures. As $2^n > n^8$ for
$n \geq 50$, this shows that P_n can have no m-directional representation at all. First we consider how many of the maximals of P_n may be represented by line segments or points.

To this end, we consider for each point p in an unbounded face, the rays centred at p. To each ray we assign a vector consisting of its successive intersections with the lines.

As there are at most $C(k,2)$ regions of intersection of these lines, there are at most $C(n,2)$ such vector sequences for each point, and for convenience, we may suppose that the rays are generated in counterclockwise order, say. We say that two such points p and p' are equivalent if the corresponding sequences of vectors are identical. In fact, two points p and p' in the same unbounded face are not equivalent just if there are two distinct points a and b, each an intersection of some pair of the lines on the plane, corresponding to the minimals, such that the points p, p' lie on opposite sides of the line joining a and b. To see this let p and p' be non-equivalent points in some unbounded face. Let the initial ray for each point be chosen toward the unbounded region, that is, the rays whose vectors are, in effect, empty, for the rays intersect no line of the collection. Now, as the rays from p and p' are rotated counterclockwise in pairs, there is a first pair of rays which have assigned different vectors. Indeed, there must then be two points a, b, each a point of intersection of pairs of lines, such that a ray from p passes through a before its twin from p' does. Then p and p' do lie on opposite sides of the line joining a and b. (See Figure 18.)

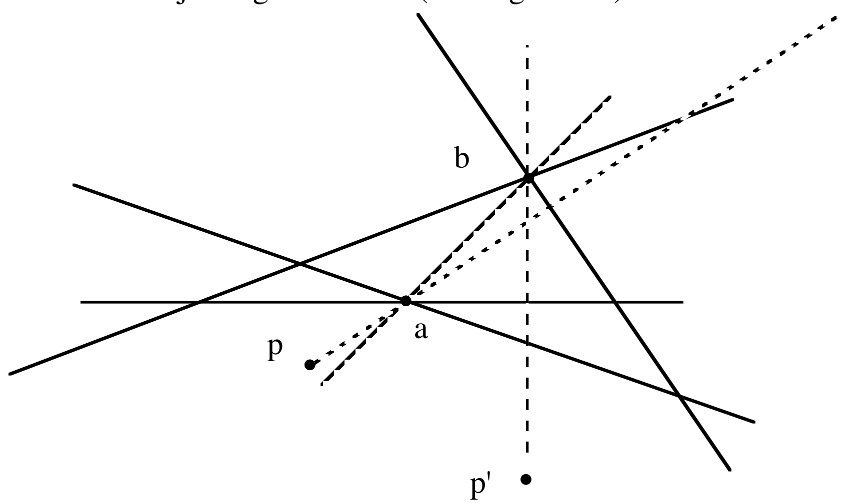


Figure 18

To estimate the maximum number of these equivalence classes, consider all points of intersection of pairs of lines corresponding to the minimals. There are C(k,2) such points which in turn, generate up to C(C(k,2),2) lines which produce 1+ (1/2) C(C(k,2),2)(C(C(k,2),2)+1)
faces. This is the maximum number of equivalence classes of points. Of these, just 2C(C(k,2),2) correspond to unbounded faces. Now, if a line segment is used to represent a maximal element of P_n, it must coincide with such a ray emanating from an unbounded face. Actually every such ray may coincide with up to C(k,2) different lines. For each ray there are at most C(k,2) distinct line segments. Thus, the maximum number of such distinct line segments, each corresponding to a different family of minimals, is 2C(C(k,2),2)C(k,2)C(k,2)= o(n^8).

Now, we estimate how many of the maximals of P_n may be represented by convex figures, which are not themselves line segments.

First we consider all such convex figures which contain a point of intersection of two lines. As the figures themselves are disjoint, there are at most C(k,2) such figures possible.

Now, consider any convex figure which does not contain an intersection point but which crosses every face in at most two sides of the face. Then there is a line segment contained in this convex figure which meets every face met by this convex figure itself. If there were a path inside this convex figure which crossed a side of a face which this line segment does not cross, then there would be a face, three sides of which are crossed by this convex figure. This means that the minimal elements of P_n which are contained in the maximal element represented by this convex figure are entirely enumerated by the line segment passing through this convex figure. Thus, all such convex figures have, in effect, already been counted by the earlier enumeration of line segments.

Finally, we consider those convex figures which contain no intersection points and which meet at least three sides of some face. As the convex figures must be disjoint, a face may correspond to at most k-3 such convex figures (a triangulation of an n-sided polygon). Enumerating over all faces, we find at most

$$(k - 3)(1 + {}^1/_2 k(k+1)) = o(n^3)$$

such convex figures.

In all, there can be at most $o(n^8)$ convex figures, although we require $o(2^n)$ of them. This completes the proof.

We do not know how far from best possible this estimate is for n (≥ 50) . At least for each
$n \leq 4$ blocking representations do exist for P_n (see Figure 19). Is there one for P_5?

As a matter of fact, we have been unable to decide whether the dual of P_n has a blocking relation, for every n. More generally, we have no counterexample at all to the

CONJECTURE: *The dual of a blocking relation is a blocking relation.*

Another interesting problem is this

CONJECTURE: *Every blocking relation can be represented using only line segments.*

(cf. R. Nowakowski, I. Rival, J. Urrutia.)

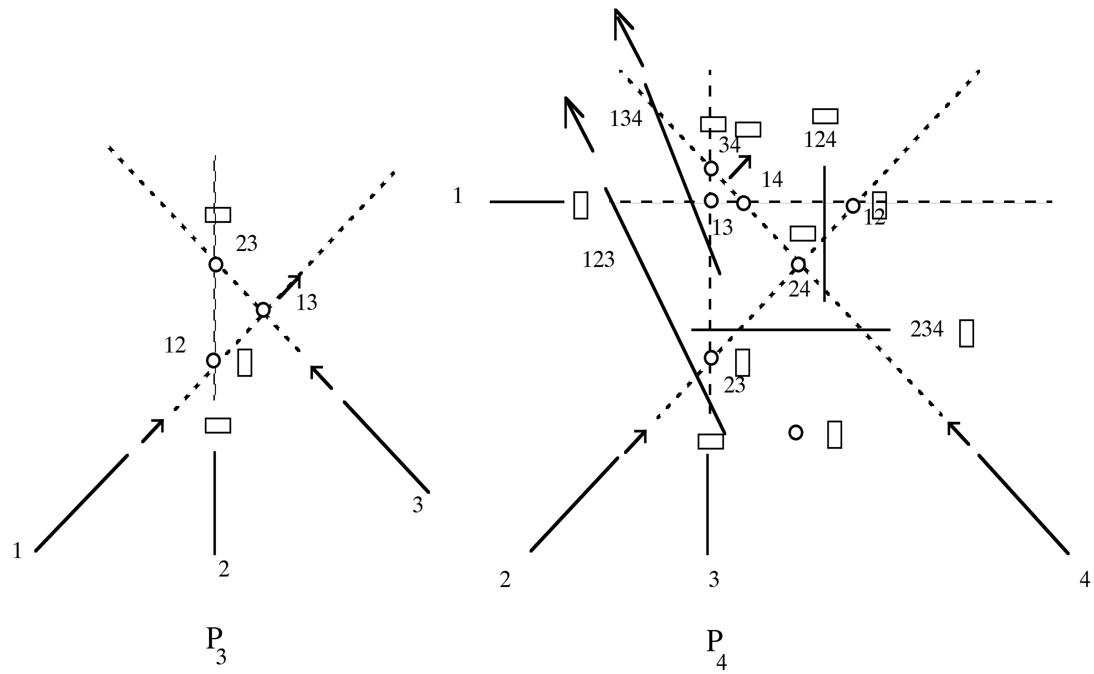


Figure 19.

Finally, we have also been unsucsesful in deciding this
CONJECTURE: *Every finite truncated lattice is a blocking relation.*

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