Sequentially Divisible Dissections of Simple Polygons

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Abstract

A k-dissection $\mathcal{D}$ of a polygon $\mathcal{P}$, is a partition of $\mathcal{P}$ into a set of subpolygons $\{\mathcal{Q}_1, \ldots, \mathcal{Q}_m\}$ with disjoint interiors such that these can be reassembled to form k polygons $\mathcal{P}_1, \ldots, \mathcal{P}_k$ all similar to $\mathcal{P}$. $\mathcal{D}$ is called *non-trivial* if none of $\{\mathcal{Q}_1, \ldots, \mathcal{Q}_m\}$ is similar to $\mathcal{P}$.

In this paper we show that any convex n-gon has a k-dissection (resp. sequential dissection) with $(k-1)n+1$ pieces, $n \leq 5$.

Let $k \geq 2$ and $n \geq 3$ be integers and let P be an n-gon. We show that if P is a convex polygon and $n \leq 5$, then there exists a dissection of P consisting of at most $(m-1)n+1$ polygons which combine to form sequentially $2, 3, \cdots, m$ unequal polygons similar to P. If P is a convex polygon and $n \geq 6$, then there exists a dissection of P consisting of at most $\lfloor \frac{5m-4}{3} n \rfloor - 2(m-1)$ polygons which can be assembled similarly as stated above. We also show that for $m \equiv 1 \pmod{3}$ and general n-gon P, we can dissect P into at most $2n - 2 + \frac{m-4}{3}(\lfloor \frac{7}{3} n \rfloor - 4)$ polygons which combine to form sequentially $4, 7, \cdots, m$ polygons similar to P.

1 Introduction

Dissections of polygons is a truly classical field of study in the mathematical sciences. A classical result of the 18-th century by Lowry, Wallace, Bolyai, and Gerwing, asserts that given two simple polygons $\mathcal{P}$ and $\mathcal{Q}$ of the

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same area, we can dissect $\mathcal{P}$ into a finite number of polygons which can be reassembled to form $\mathcal{Q}$.

Books on dissections of polygons appear from time to time in the litterature, each bringing new advances and interesting puzzles to the topic we study here, e.g. Fourrey [3](1907), Kraitchik [6](1942), Gardner [5](1961), $\ldots$, and lately Frederickson [4].

Let $\mathcal{P}$ be a polygon on the plane. A *k-dissection* $\mathcal{D}$ of $\mathcal{P}$ is a partitioning of $\mathcal{P}$ into subpolygons $\{\mathcal{P}_1, \ldots, \mathcal{P}_m\}$ with disjoint interiors such that they can be reassembled to form k polygons all similar to $\mathcal{P}$. Each $\mathcal{P}_i$ is called a *piece* of $\mathcal{D}$. If none of the pieces of $\mathcal{D}$ is similar to $\mathcal{P}$, $\mathcal{D}$ is called *non-trivial*. A dissection of $\mathcal{P}$ is called *sequentially k-divisible* if for every j, $1 \leq j \leq k$, its pieces can be assembled so as to form j polygons similar to $\mathcal{P}$. In Figure 1(a), we show a sequentially 2-divisible dissection of a triangle. Figure 1(b) shows a non-trivial sequentially 2-divisible dissection of the same triangle.

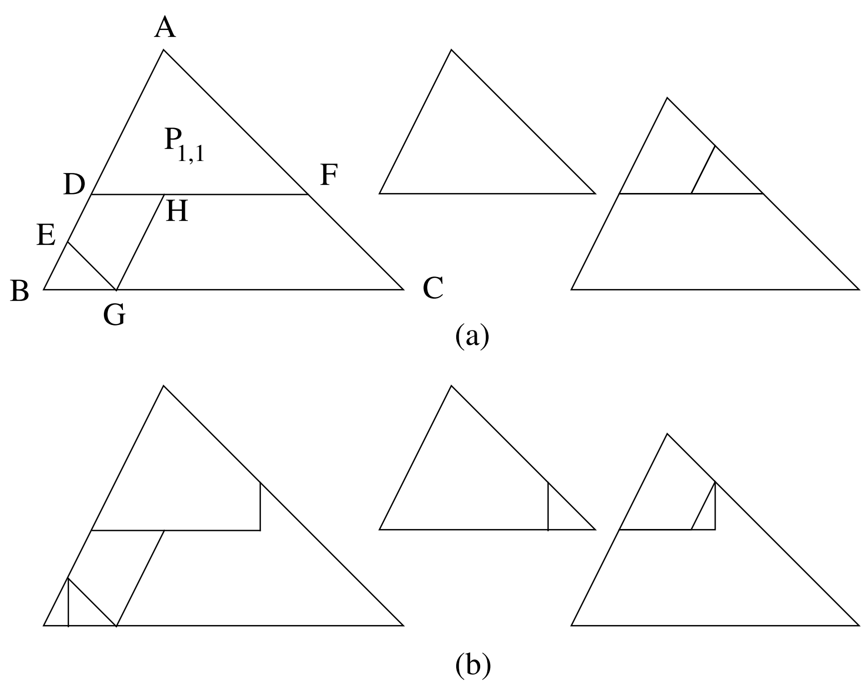


Figure 1: A 2-dissection and a nontrivial 2-dissection of a triangle.

Sequentially k-divisible dissections of squares have been studied in [1, 2, 7, 8]. In this paper we present sequentially k-divisible dissections of triangles, convex quadrilaterals, and convex pentagons with $3k-2$, $4k-3$, and $5k-4$ pieces respectively. For triangles we present non-trivial sequentially k-divisible dissections with $3k-1$ pieces. For regular $4n$-gons, we present sequentially k-divisible dissections with $(k-1)n+1$ or $(k-1)n-k+2$ pieces for n odd and even resp. Finally for *simple polygons*, not necessarily convex, with n vertices we present a 4-dissection with $2n-2$ pieces. This allows us to construct $4+3k$-dissections with at most $(2n-2)+k(2n+\lfloor\frac{n}{3}\rfloor-4)$ pieces.

2 Sequentially divisible dissections of triangles

Two polygons $\mathcal{P}$ and $\mathcal{Q}$ are called similar if there is mapping $f:\mathbf{R}^2\to\mathbf{R}^2$ such that $f(x)=p_0+\lambda x$, and $f(\mathcal{P})=\mathcal{Q}$, where p_0 is a point in $\mathbf{R}^2$. $\mathcal{P}$ and $\mathcal{Q}$ are called congruent if there is a translation T, a rotation R, and perhaps a reflection that maps $\mathcal{P}$ onto $\mathcal{Q}$.

Let $\mathcal{P}$ be a polygon on the plane. A dissection $\mathcal{D}$ of $\mathcal{P}$ is a partititoning of $\mathcal{P}$ into m subpolygons $\mathcal{P}_1,\ldots,\mathcal{P}_m$ such that $int\mathcal{P}_i\cap int\mathcal{P}_j=\phi$, $1\le i<j\le m$, where $int\mathcal{P}$ denotes the interior of $\mathcal{P}$. Each $\mathcal{P}_i$ is called a *piece* of $\mathcal{D}$. Given two polygons $\mathcal{P}$ and $\mathcal{Q}$ we say that $\mathcal{P}$ can be dissected into $\mathcal{Q}$ if there are dissections $\mathcal{D}=\{\mathcal{P}_1,\ldots,\mathcal{P}_m\}$ and $\mathcal{D}'=\{\mathcal{Q}_1,\ldots,\mathcal{Q}_m\}$ of $\mathcal{P}$ and $\mathcal{Q}$ such that $\mathcal{P}_i$ is congruent to $\mathcal{Q}_i$, $i=1,\ldots,m$. We will also say that the pieces of $\mathcal{D}$ can be reassembled into $\mathcal{Q}$. If none of the pieces of $\mathcal{D}$ is similar to $\mathcal{P}$, $\mathcal{D}$ will be called a *non-trivial* dissection of $\mathcal{P}$.

The following notation will be useful throughout our paper: Given two points P and Q on the plane, PQ will denote the line segment joining them. The point $(1-\lambda)P+\lambda Q$ will be denoted as $\lambda(PQ)$. Notice that $\lambda(PQ)$ is different from $\lambda(QP)$. For example when $\lambda=0$, we obtain P, and when $\lambda=1$ we get Q, and when $\lambda=\frac{1}{2}$ we obtain the mid point of the segment PQ. In a similar way, let $\mathcal{Q}$ be a polygonal with vertices $Q_1,\ldots,Q_n$, then $\lambda(P,\mathcal{Q})$ will denote the poligonal with vertices $\lambda(PQ_1),\ldots,\lambda(PQ_n)$. If line segments PQ and RS are parallell, we will write $PQ\parallel RS$. If a polygon $\mathcal{P}$ has vertices $V_1,\ldots,V_n$ we will often refer to it as the polygon $\{V_1,\ldots,V_n\}$.

Observe that if two polygons $\mathcal{P}$ and $\mathcal{Q}$ are similar, any dissection $\mathcal{D}=\{\mathcal{P}_1,\ldots,\mathcal{P}_m\}$ of $\mathcal{P}$ induces in a natural way, a dissection $\mathcal{D}'$ of $\mathcal{Q}$ such that the pieces of $\mathcal{D}'$ are the sets $f(\mathcal{P}_i)$, $i=1,\ldots,m$. Dissection $\mathcal{D}'$ will be refered to as the *dissection induced* in $\mathcal{Q}$ by $\mathcal{D}$. We now prove:

Theorem 2.1 *Any triangle has a sequentially k-divisible dissection (resp. non-trivial sequentially k-divisible dissection) with $3k-2$ pieces (resp. $3k-1$).*

Proof: Let $\mathcal{P}_{0,1}$ be a triangle with vertices $\{A, B, C\}$, and assume that the perpendicular line through A to the line segment BC intersects it. Let $\mathcal{D}_0$ be the dissection of $\mathcal{P}_{0,1}$ obtained as follows: Let D and E be the points on AB such that $D = \frac{3}{5}(AB)$ and $E = \frac{4}{5}(AB)$, and let $F \in AC$, $G \in BC$ and $H \in DF$ be the points such that $DF \parallel BC$, $EG \parallel AC$ and $GH \parallel AB$ (Figure 2.1.(a)). Then triangle $\mathcal{P}_{1,1}$ with vertices $\{A, D, F\}$ is similar to $\mathcal{P}_{0,1}$ and their ratio of similitude is $\frac{3}{5}$. Notice that triangle $\mathcal{P}_{1,2}$ with vertex set $\{B, E, G\}$ and trapezoids $\mathcal{P}_{1,3}$ and $\mathcal{P}_{1,4}$ with vertices $\{D, E, G, H\}$, and $\{C, F, H, G\}$ respectively can be assembled into a triangle similar to $\mathcal{P}_{0,1}$ with ratio of similitude equal to $\frac{4}{5}$, see Figure 1(a).

In a recursive way, let $\mathcal{D}_j$ be the disection induced in $\mathcal{P}_{j,1}$ by $\mathcal{D}_0$, where $\mathcal{P}_{j,1}$ is the triangle of $\mathcal{D}_{j-1}$ containing vertex A, $j \geq 1$. For any fixed integer k, $(\mathcal{D}_0 - \{\mathcal{P}_{1,1}\}) \cup (\mathcal{D}_1 - \{\mathcal{P}_{2,1}\}) \cup \ldots \cup \mathcal{D}_k$ defines a disection $\mathcal{D}^k$ of $\mathcal{P}_{0,1}$ with exactly $3k + 4$ pieces. See Figure 2(a). Clearly $\mathcal{D}^k$ is a sequentially $(k+2)$-divisible dissection of $\mathcal{P}_{0,1}$.

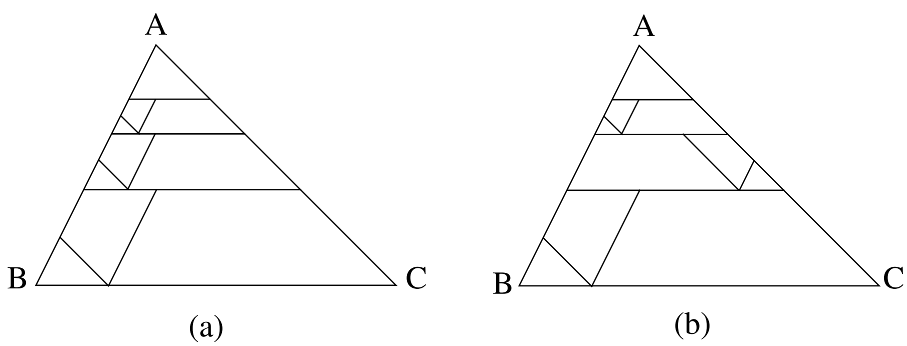
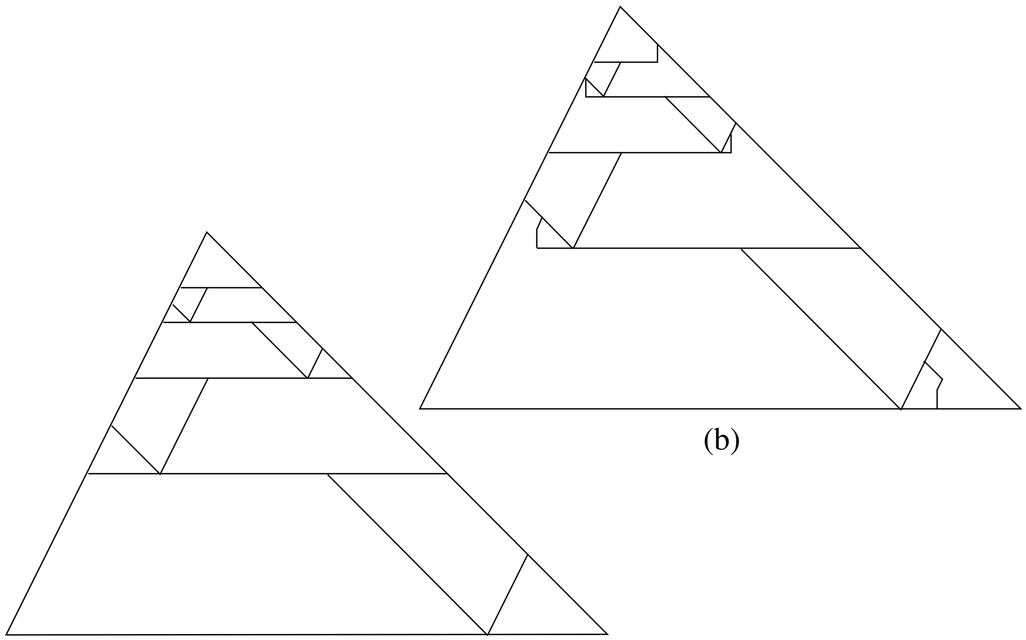


Figure 2: A sequentially 4-divisible dissection of a triangle.

Finding non-trivial sequentially k-divisible dissections of triangles is more challenging. First we start by modifying $\mathcal{D}^k$ to obtain a sequentially k-divisible dissection $\mathcal{D}$ of triangle $\{A, B, C\}$ as shown in Figure 2(b). The main objective of our modification, is to make sure that every second triangle of $\mathcal{D}^k$ from top to bottom touches AB, and the others touch AC (with the exception the triangle containing A, which touches AB, and AC). The details of this modification are straightforward, and are left to the reader. We now proceed to show how we can modify this construct to obtain a sequentially k-divisible dissection of our triangle.

Suppose that we relabel the triangles of $\mathcal{D}$ from top to bottom by



(a)

(b)

Figure 3: Finding a non-trivial sequentialy 5-dissection dissection of a triangle.

$\mathcal{T}_1, \ldots, \mathcal{T}_n$ as shown in Figure 3(a). Split $\mathcal{T}_2$ into two triangles $\mathcal{L}_2$ and $\mathcal{R}_2$ by drawing a vertical through its top vertex. Join triangle $\mathcal{L}_2$, the left piece of $\mathcal{T}_2$ to the element of $\mathcal{D}$ below it. Next split $\mathcal{T}_1$ into two pieces, one of which, containing its rightmost vetex, is similar to $\mathcal{R}_2$, and join the right piece to the element of $\mathcal{D}$ below it, as shown in Figure 3(b). In a recursive way, we now split $\mathcal{T}_i$ into a right piece $\mathcal{R}_i$ and a left piece $\mathcal{L}_i$ such that if i is odd, then $\mathcal{L}_i$ is congruent with $\mathcal{L}_{i-1}$, and if i is even then $\mathcal{R}_i$ is congruent to $\mathcal{R}_{i-1}$, $i = 2, \ldots, k$. Next if i is odd, join $\mathcal{R}_i$ to the piece of $\mathcal{D}$ below it, else if i is even join $\mathcal{L}_i$ to the piece of $\mathcal{D}$ below it, $i < k$, see Figure 3(b). It is now easy to see that the pieces of the dissection thus obtained form a sequentially k-divisible dissection. In Figure 4 we show how to assemble the pieces of the dissection in Figure 3(b) into five triangles.

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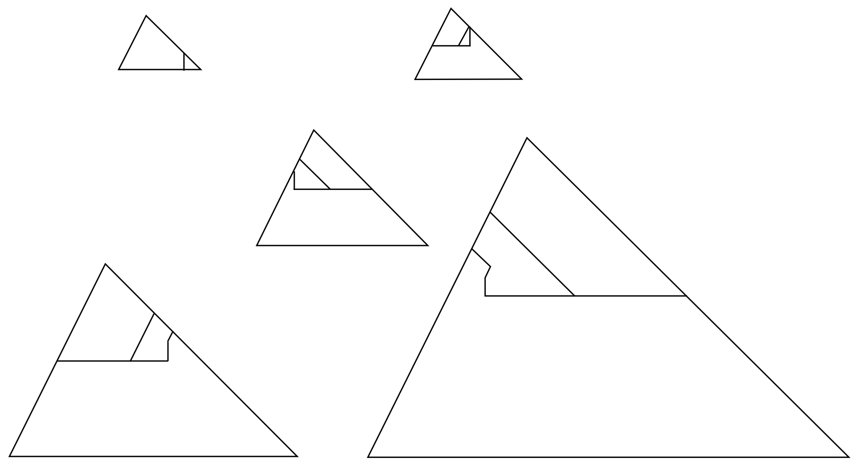

Figure 4: Reassembling the dissection in Fig 3(b) into five triangles.

3 Quadrilaterals

We now show:

Theorem 3.1 *Any convex quadrilateral has a sequentially k-divisible dissection with $4k-3$ pieces.*

Proof: Let $\mathcal{P}=\{A,B,C,D\}$ be a convex quadrilateral to be dissected. Since $(\angle A+\angle B)+(\angle C+\angle D)=2\pi$, we may assume that $\angle A+\angle B\leq\pi$. Since $(\angle A+\angle D)+(\angle B+\angle C)=2\pi$, we may also assume that $\angle B+\angle C\leq\pi$.

We first give a sequentially 2-divisible dissection of $\mathcal{P}$ consisting of five pieces. Let $E\in AB$, $F\in AD$ be the points such that $E=\frac{3}{5}(AB)$, $F=\frac{3}{5}(AD)$ and let G be the point on the diagonal AC such that $EG\parallel BC$ (so $FG\parallel DC$). Let $H\in BC$, $I\in DC$ be the points such that $H=\frac{1}{5}(BC)$, $I=\frac{1}{5}(DC)$, let $J\in EG$, $K\in FG$ be the points such that $JH\parallel AB$ and $KI\parallel AD$ and let Ł be the mid-point of JH. Let M be the intersection point of the line passing through E and parallel to DC and the line passing through L and parallel to AD. Since $\angle A+\angle B\leq\pi$ and $\angle B+\angle C\leq\pi$, M is a point in the parallelogram $\{B,E,J,H\}$. This produces the the dissection of $\mathcal{P}$ with pieces $\{\mathcal{P}_{1,1},\ldots,\mathcal{P}_{1,5}\}$ as shown in Figure 5(a). It is easy to see now that this is a sequentially 2-divisible dissection of $\mathcal{P}$ (Figure 5(b)).

In a recursive way let $\mathcal{D}_i$ be the dissection induced in $P_{i,1}$ by $\mathcal{D}$. This produces a sequence of sequentially k-divisible dissections of $\mathcal{P}$ with $4k-3$ pieces. ■

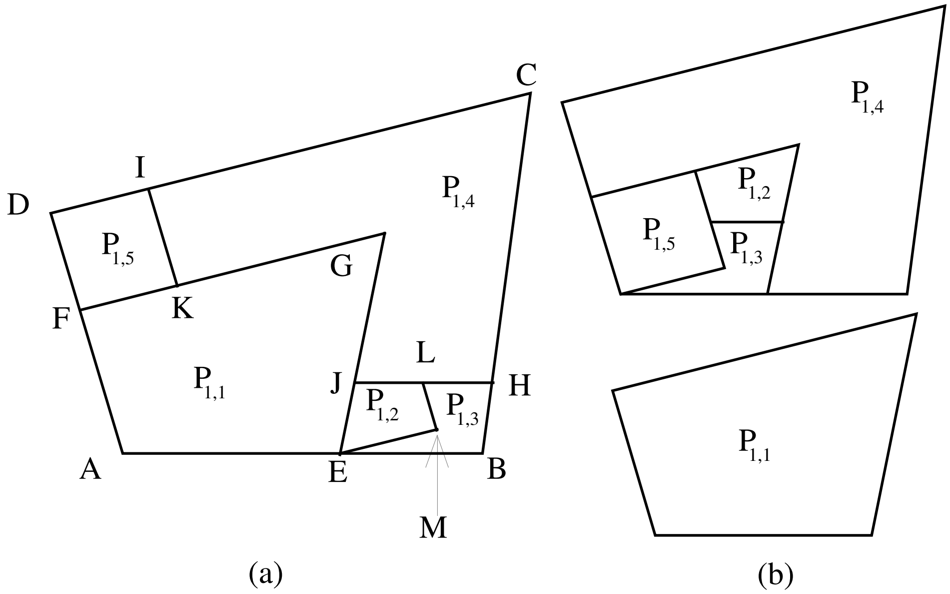

Figure 5: A 2-dissection of a convex cuadrilateral.

4 Pentagons

Theorem 4.1 *Any convex pentagon has a sequentially k-divisible dissection with $5k-4$ pieces.*

Let $\mathcal{P}$ be a pentagon. As in the previous section, we will exibit a 2-sequential dissection of $\mathcal{P}$ into six pieces $\mathcal{P}_1, \ldots, \mathcal{P}_6$ such that:

1. $\mathcal{P}_1$ is similar to $\mathcal{P}$
2. $\mathcal{P}_2, \ldots, \mathcal{P}_5$ can be assembled into a pentagon similar to $\mathcal{P}$.

Some preliminary results will be proved now.

Lemma 4.2 *Let $\mathcal{P}$ be a pentagon. Then we can label its vertices A, B, C, D, and F in the clockwise or counter-clockwise direction such that:*

1. $\angle A + \angle B > \pi$, $\angle B + \angle BCE \geq \pi$
2. *At least one of the following holds:*
   $\angle ACD + \angle D \geq \pi$ *or* $\angle D + \angle E > \pi$

Proof: Assume that the vertices of $\mathcal{P}$ are labelled $P_1, \ldots, P_5$ in the counterclockwise direction, and for each i let $\mathcal{Q}_i$ be the polygon with vertices $\{P_1, \ldots, P_5\} - \{P_{i+2}\}$, addition taken mod 5. If the sum of the angles of $\mathcal{Q}_i$ at P_i and P_{i+1} is greater than π we color P_i with color 0, else color P_i with color 1.

It now follows that there is an index i such that one of the following two conditions hold:

1. P_i, and P_{i+1} are colored 0 and P_{i-1} is colored 1 or P_{i-2} is colored 0
2. P_i, and P_{i+1} are colored 1, and P_{i+2} is colored 0 P_{i+3} is colored 1

It is now easy to verify that, in the first case when P_i and P_{i+1} are colored 0, if P_{i-1} is colored 1, then $\angle P_i + \angle P_{i+1} > \pi$, $\angle P_{i+1} + \angle P_{i+1}P_{i+2}P_{i+4} \geq \pi$ and $\angle P_i P_{i+2} P_{i+3} + \angle P_{i+3} \geq \pi$. If P_{i-2} is colored 0, then $\angle P_i + \angle P_{i+1} > \pi$, $\angle P_{i+1} + \angle P_{i+1}P_{i+2}P_{i+4} \geq \pi$, and $\angle P_{i+3} + \angle P_{i+4} > \pi$. Let $A = P_i, \ldots, E = P_{i+4}$.

The case when P_i and P_{i+1} are colored 1 is solved in a similar way, with $A = P_i,\ B = P_{i-1}, \ldots,\ E = P_{i-4} = P_{i+1}$.

■

Let $\mathcal{P}$ be a pentagon with its vertices labelled A, B, C, D, E as in Lemma 4.2. Let $A_1 = A$, $B_1 = \frac{3}{5}(AB)$, $\ldots$,$E_1 = \frac{3}{5}(AE)$, $A_2 = \frac{2}{5}(DA)$,$\ldots$, $C_2 = \frac{2}{5}(DC)$, $D_2 = D$, and $E_2 = \frac{2}{5}(DE)$. Let $F_1 = \frac{2}{5}(BC)$, and $G_1 = \frac{1}{3}(B_1C_1)$.

Since $\angle B + \angle C > \pi$ by the first condition 1 in Lemma 4.2, pentagons $\{A_1, B_1, C_1, D_1, E_1\}$ and $\{A_2, B_2, C_2, D_2, E_2\}$ have no common inner point. Notice that BB_1 and B_2A_2 have the same length. Since $\angle A + \angle B > \pi$ the translation that maps BB_1 to B_2A_2 maps the rectangle $\{B, F_1, G_1, B_1\}$ to a subset of the pentagon $\{A_2, B_2, C_2, D_2, E_2\}$.

Consider next the pentagon with vertices $C_3 = C_2,\ A_3 = \frac{1}{2}(C_2A_2),\ B_3 = \frac{1}{2}(C_2B_2),\ D_3 = \frac{1}{2}(C_2D_2),\ E_3 = \frac{1}{2}(C_2E_2)$. Let us now rotate pentagon $\{A_3, B_3, C_3, D_3, E_3\}$ 180 degrees around A_3 as shown in Figure 6(a) to obtain the pentagon $\mathcal{P}_4$ with vertices $\{A_4, B_4, C_4, D_4, E_4\}$.

Two cases arise: E_4 belongs to the interior of pentagon $\{A, B, C, D, E\}$ as in Fig 6(a), or E does not lie in the interior of the same pentagon.

In the first case, the reader can now verify that the dissection shown in Figure 7(a) is realizable. In the case when E_4 lies outside of our original pentagon, (this situation can arise if we move point E in Figure 6(a) far enough to the right, for aesthetic reasons we don't show a picture for this case) we will show that the pentagon obtained by translating $\{A_3, \ldots, E_3\}$

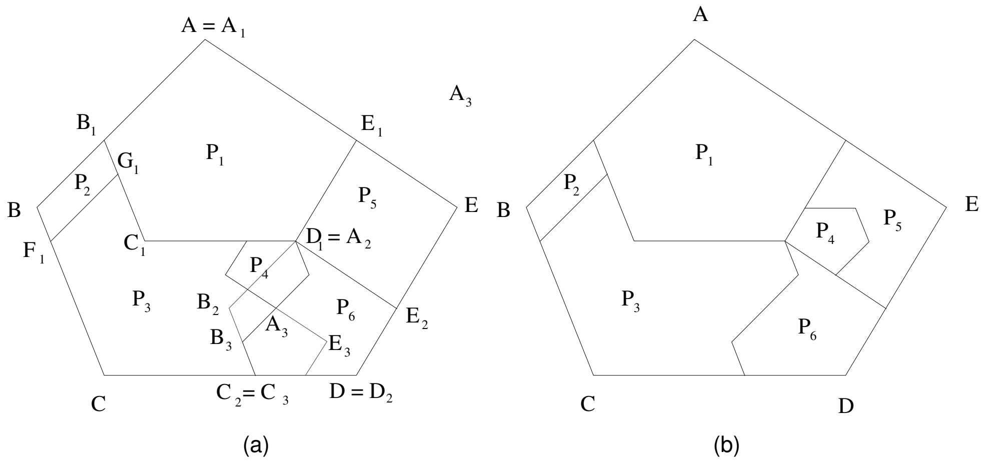

Figure 6: Finding a dissection of a convex pentagon.

such that E_3 lies on D_1 is contained in the parallelogram $\{D_1, E_2, E, E_1\}$. The dissection shown in Figure 7(b) will now be realizable.

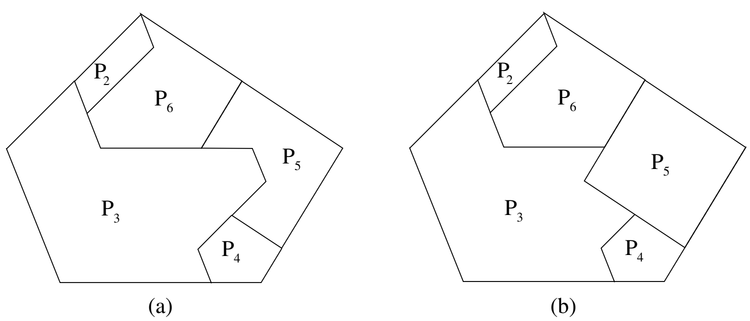

Figure 7: Showing the regrouping of the pieces of the dissection from figure 6. P_1 is not shown here, but it is a part of the 2-dissections illustrated here.

Given two points P and Q, $\overrightarrow{PQ}$ will denote the vector $P - Q$. Let $\boldsymbol{a} = \overrightarrow{AC}$, $\boldsymbol{b} = \overrightarrow{AE}$ and consider the unique real numbers α and β such that

$$\overrightarrow{AD} = \alpha \boldsymbol{a} + \beta \boldsymbol{b}.$$

Clearly

$$\alpha > 0,\ \beta > 0 \text{ and } \alpha + \beta > 1 \tag{4.1}$$

We also have

$$\beta \le 1 \text{ or } \alpha < 1, \tag{4.2}$$

depending on whether $\angle ACD + \angle D \ge \pi$ or $\angle D + \angle E > \pi$ holds.

Two cases arise:

Case 1. $\beta > \max\{\frac{1}{3},\, \alpha - 1\}$:

This corresponds to the case when E_4 belongs to the interior of our pentagon. For this purpose, we show that $\mathcal{P}_4 \subset \{F_1,\ C,\ C_2,\ F_2,\ B_4,\ C_4,\ C_1, G_1\}$ and that $\{F_1,\ C,\ C_2,\ F_2,\ B_4,\ C_4,\ C_1,\ G_1\} \setminus \mathcal{P}_4$ is connected. Since

$$\begin{aligned}
\overrightarrow{C_4E_4} &= -\tfrac{1}{5}\overrightarrow{CE} \\
&= \tfrac{1}{5}(\boldsymbol{a} - \boldsymbol{b}) \\
&= \tfrac{1}{3\beta}\cdot\tfrac{3}{5}[\boldsymbol{a} - (\alpha\boldsymbol{a} + \beta\boldsymbol{b})] + \tfrac{\alpha+\beta-1}{2\beta}\cdot\tfrac{2}{5}\boldsymbol{a} \\
&= \tfrac{1}{3\beta}\overrightarrow{C_4C_1} + \tfrac{\alpha+\beta-1}{2\beta}\overrightarrow{C_4C_2}
\end{aligned}$$

and since $0 < \frac{1}{3\beta} < 1$ and $0 < \frac{\alpha+\beta-1}{2\beta} < 1$ by the assumption of Case 1 and (4.1), E_4 is an inner point of the parallelogram $\{C,\ C_2,\ C_4,\ C_1\}$. Since $\angle B_4A_4E_4 < \pi$, this implies $E_4 \in \operatorname{int}\{F_1,\ C,\ C_2,\ F_2,\ B_4,\ C_4,\ C_1,\ G_1\}$, and hence $\mathcal{P}_4 \subset \{F_1,\ C,\ C_2,\ F_2,\ B_4,\ C_4,\ C_1,\ G_1\}$ and $\{F_1,\ C,\ C_2,\ F_2,\ B_4,\ C_4,\ C_1,\ G_1\}\setminus \mathcal{P}_4$ is connected, as desired.

Case 2. $\beta \le \max\{\frac{1}{3},\, \alpha - 1\}$:

We will show now that in this case $\mathcal{P}_5 \subset \{E_1,\ E_5,\ E_2,\ E\}$ and that $\{E_1,\ E_5,\ E_2,\ E\} \setminus \mathcal{P}_5$ is connected. This will prove that the dissection shown in Figure XXX is realizable. By (4.1), (4.2) and the assumption of Case 2 we have that

$$\alpha > \tfrac{2}{3} \text{ and } \beta < 1 \tag{4.3}$$

Let γ and δ be unique real numbers such that $\overrightarrow{EB} = \gamma\overrightarrow{EA} + \delta\overrightarrow{EC}$ $(= \gamma(-\boldsymbol{b}) + \delta(\boldsymbol{a} - \boldsymbol{b}))$.

Since $\angle A + \angle B > \pi$, $\angle B + \angle BCE \ge \pi$ we have that:

$$0 < \delta \le 1 \text{ and } 0 < \gamma < 1 \tag{4.4}$$

We also have

$$\begin{aligned}
\overrightarrow{E_5B_5} &= -\tfrac{1}{5}\overrightarrow{EB} \\
&= -\tfrac{1}{5}[\gamma(-\boldsymbol{b}) + \delta(\boldsymbol{a} - \boldsymbol{b})] \\
&= \tfrac{\delta}{3\alpha}\cdot\tfrac{3}{5}[\boldsymbol{b} - (\alpha\boldsymbol{a} + \beta\boldsymbol{b})] + \tfrac{(\gamma+\delta)\alpha+\delta(\beta-1)}{2\alpha}\cdot\tfrac{2}{5}\boldsymbol{b} \\
&= \tfrac{\delta}{3\alpha}\overrightarrow{E_5E_1} + \tfrac{(\gamma+\delta)\alpha+\delta(\beta-1)}{2\alpha}\overrightarrow{E_5E_2},
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
\overrightarrow{E_5C_5} &= -\tfrac{1}{5}\overrightarrow{EC} \\
&= -\tfrac{1}{5}(\boldsymbol{a}-\boldsymbol{b}) \\
&= \tfrac{1}{3\alpha}\cdot\tfrac{3}{5}[\boldsymbol{b}-(\alpha\boldsymbol{a}+\beta\boldsymbol{b})] + \tfrac{\alpha+\beta-1}{2\alpha}\cdot\tfrac{2}{5}\boldsymbol{b} \\
&= \tfrac{1}{3\alpha}\overrightarrow{E_5E_1} + \tfrac{\alpha+(\beta-1)}{2\alpha}\overrightarrow{E_5E_2}. \qquad (4.6)
\end{aligned}$$

Since $0 < \frac{\delta}{3\alpha} < \frac{1}{3\alpha} < \frac{1}{2}$, $0 < \frac{(\gamma+\delta)\alpha+\delta(\beta-1)}{2\alpha} < \frac{2\alpha+(\beta-1)}{2\alpha} < 1$ and $0 < \frac{\alpha+(\beta-1)}{2\alpha} < \frac{1}{2}$ by (4.3) and (4.4), it follows from (4.5) and (4.6) that B_5 and C_5 are inner points in the parallelogram $\{E_1,\ E_5,\ E_2,\ E\}$. Hence $\mathcal{P}_5 \subset \{E_1,\ E_5,\ E_2,\ E\}$ and $\{E_1,\ E_5,\ E_2,\ E\} \setminus \mathcal{P}_5$ is connected, as desired.

Consider next any of the dissections $\mathcal{D}$ shown in Figure 6. As we did before, we will now take the dissection induced by $\mathcal{D}$ on P_1 to obtain a 3-dissection of $\mathcal{P}$. By iterating this process, we get a sequence of sequentially k-divisible dissections of $\mathcal{P}$ with $5k-4$ pieces.

5 Hexagons

In Figure 8 we give a 2-sequential dissection $\mathcal{D}$ of a *regular hexagon* P with vertices A, B, C, D, E, F into 7 pieces. In this figure, $B_1 = \frac{3}{5}(AB)$, $C_1 = \frac{3}{5}(AC)$, $W = \frac{1}{3}(B_1C_1)$, and $Y = \frac{3}{5}(CD)$. Also the following distances are one fifth of the distance from B to C: the distance from B_1 to W, the distance from X to Y, and the distance from D_1 to Z. The remaining details are left to the reader. By recursively using the dissection induced in P_1 by $\mathcal{D}$ it follows that there are k-sequential dissections of regular hexagons with $6m-5$ pieces.

6 Sequentially divisible dissections of regular 4k-gons

In page 97 of [4], 2-dissections of regular n-gons are given. For n even those dissections contain n pieces, for n odd, $n+1$ pieces.

Using the iteration process studied in the second section of our paper the following result now follow:

Theorem 3.A. Let $\mathcal{P}$ be a regular polygon with n vertices, and $k \geq 2$ be an integer. Then if n is odd there is a sequentially k-divisible dissection

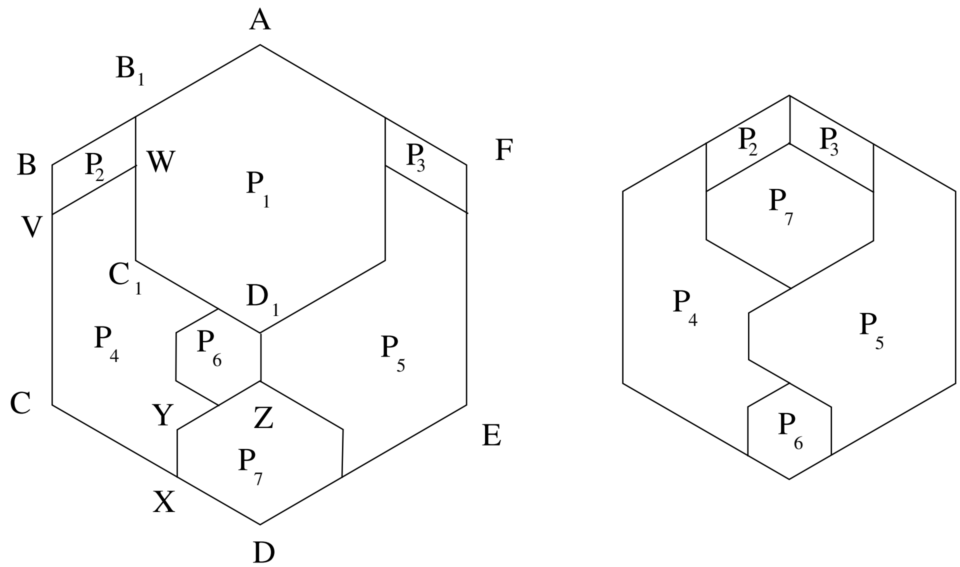

Figure 8: A 2-dissection of a regular hexagon. We show how to reassemble $\{P_2, \ldots, P_7\}$ to form a regularhexagon.

of $\mathcal{P}$ with $(k-1)n+1$ pieces. If n is even a sequentially k-divisible dissection with $(k-1)n-k+2$ pieces exists.

We now give a new 2-dissection of regular $4k$-gons with $4k$ pieces. Let $\mathcal{P}$ be a regular polygon with $4m$ vertices labelled $A_0, \ldots, A_{4m-1}$ in the counterclockwise direction, with A_0 being the topmost vertex of $\mathcal{P}$.

Consider a second regular $4m$ polygon $\mathcal{B}$ with vertices $B_0, \ldots, B_{4m-1}$ of size $\frac{2}{5}$ that of $\mathcal{P}$. We present first a dissection of $\mathcal{P}_2$ with $2m$ pieces obtained as follows:

For each i, $1 \leq i \leq m$ let $\mathcal{Q}_i$ be the polygonal with vertices $B_0, \ldots, B_{2i}$, and let $\mathcal{Q}_i^{`}$ be the polygonal $\frac{1}{2}(B_{2i}, \mathcal{Q}_i)$. Let $\mathcal{Q}_{i,1}$ be the polygonal obtained by joining a copy of $\mathcal{Q}_i^{`}$ with the polygonal obtained by rotating $\mathcal{Q}_i^{`}$ 180 degrees around the point $\frac{1}{2}(B_{2i}B_0)$ as shown in Figure . Finally let $\mathcal{Q}_{i,2}$ be the polygonal obtained from $\mathcal{Q}_{i,1}$ by rotating it 180 degrees around the center of $\mathcal{B}$. The set of $\mathcal{Q}_{i,1}$, $\mathcal{Q}_{i,2}$, $i = 1, \ldots, m$ induces a partitioning of $\mathcal{B}$ into $2m$ pieces. Let us label the regions of this partitioning $\mathcal{P}_{2i}$, $i = 1, \ldots, 2m$ as shown in Figure for the case $m = 3$. With this labelling $\mathcal{P}_{2i}$ will contain vertex B_{2i-1} of $\mathcal{B}$.

We now show a dissection of $\mathcal{P}$ that will contain pieces similar to $\mathcal{P}_{2i}$, $i = 1, \ldots, 2m$. Let $\mathcal{P}_1$ be the polygon $\frac{3}{5}(A_0, \mathcal{P})$. Le us label the vertices of $\mathcal{P}_1$ by C_i, $i = 0, \ldots, 4m-1$, where $C_0 = A_0$.

For each $1 \leq i \leq m$ let us translate a copy of $\mathcal{P}_{2i}$ so that vertex B_{2i-1} is mapped to vertex A_{2i-1} of $\mathcal{P}$. Since the lenght of segment $A_{2i-1}C_{2i-1}$ is $\frac{2}{5}$ the lenght of $A_{2i-1}A_0$ the point B_0 of $\mathcal{P}_{2i}$ maps to vertex C_{2i-1} of $\mathcal{P}_1$. See Figure . We now flip our current construction along the line passing through A_0 and A_{2m}, to obtain a dissection of $\mathcal{P}$ with $4m$ pieces as shown in Figure TOCOME. Label the immages of P_{2i} under our flipping along the line determined by A_0 and A_{2m} by $\mathcal{P}_{4m-2i+2}$, $i = 1, \ldots, m$ as shown in the same figure. Clearly when we reflect $\mathcal{P}_{4m-2i+2}$, $i = 1, \ldots, m$, the resulting pieces togeter with $\mathcal{P}_{2i}$, $i = 1, \ldots, m$ can be reassembled to form $\mathcal{B}$. It is now easy to verify that the remaining pieces, $\mathcal{P}_3, \ldots, \mathcal{P}_{4m-1}$ can be assembled to form a polygon similar to $\mathcal{P}$ of size $\frac{4}{5}$ the size of $\mathcal{P}$ minus a polygon congruent to $\mathcal{B}$. It now follows that the resulting partititoning of $\mathcal{P}$ is a 2-dissection of $\mathcal{P}$.

7 Dissecting simple polygons

Consider a simple polygon $\mathcal{P}$ with n vertices. We now present a 4-dissection of $\mathcal{P}$ that uses exactly $2n-2$ pieces. A triangulation T of $\mathcal{P}$ is a partition of $\mathcal{P}$ into $n-2$ triangles $\{t_1, \ldots, t_{n-2}\}$ with disjoint interiors obtained by cutting $\mathcal{P}$ along $n-3$ diagonals joining pairs of vertices of $\mathcal{P}$, see Figure 9(a). We observe now that if we dissect each $t_i \in T$ into 4 similar triangles $\{t_{i,1}, t_{i,2}, t_{i,3}, t_{i,4}\}$ by cutting it along the line segments joining the mid points of its edges we obtain a dissection $\mathcal{D}'$ of $\mathcal{P}$ with $4(n-2)$ triangles, see Figure 9(b). Clearly for each j, the set of triangles $\{t_{1,j}, \ldots, t_{n-2,j}\}$ can be reassembled to obtain 4 polygons similar to $\mathcal{P}$, $j = 1, \ldots, 4$.

We now show how to modify $\mathcal{D}'$ to obtain a 4-dissection $\mathcal{D}$ of $\mathcal{P}$ with $2n-2$ pieces. First we color the vertices of $\mathcal{P}$ with 3 colors 1, 2, and 3 such that if two vertices of $\mathcal{P}$ are adjacent in T, (i.e. that are connected by a diagonal of T or an edge of $\mathcal{P}$) they receive different colors. See Figure 10(a). Our new dissection $\mathcal{D}$ is now obtained from $\mathcal{D}'$ by eliminating the cuts made in $\mathcal{P}$ along the diagonals used to obtain T, see Figure 10(b). The pieces of $\mathcal{D}$ are n polygons each of which contains exactly one vertex of $\mathcal{P}$, plus a set of triangles, one for each $t_i \in T$. Since T contains $n-2$ triangles, it now follows that the number of pieces of $\mathcal{D}$ is exactly $2n-2$.

We now show how to assemble the pieces of $\mathcal{D}$ into four polygons similar

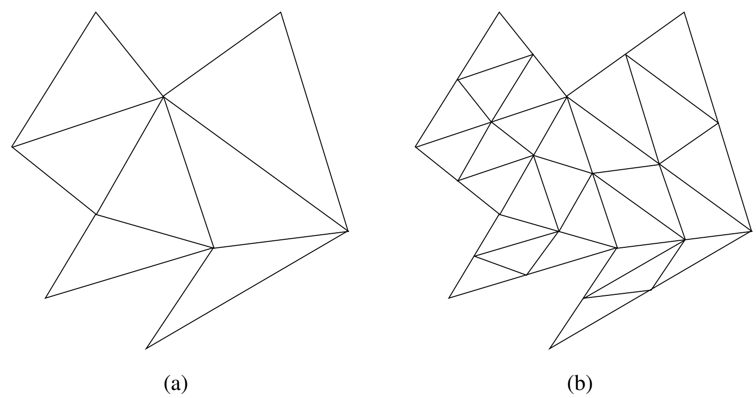

Figure 9: Triangulating and dissecting $\mathcal{P}$.

to $\mathcal{P}$. Consider the triangulation T together with the 3-vertex coloring defined before. Notice that each triangle in T has exactly one vertex of each color. For each such vertex v_i of $\mathcal{P}$ let $\mathcal{P}_i$ be polygon obtained by joining the set of triangles in T having v_i as one of its vertices. We observe now that each of the sets $S_j = \{\mathcal{P}_i : v_i \textit{ has color } j\}$, $j = 1,\ 2,\ 3$ induces a dissection of $\mathcal{P}$. Furthermore, observe that for each v_i the polygon of $\mathcal{D}$ containing it, denoted by $\mathcal{P}_i^{'}$ is similar to $\mathcal{P}_i$. It now follows that the sets of polygons $S_j^{'} = \{\mathcal{P}_i^{'} : v_i \textit{ has color } j\}$ can be reassembled to form polygons similar to $\mathcal{P}$, $j = 1,\ 2,\ 3$. Observe now that the remaining triangles of $\mathcal{D}$ can also be reassembled to form a fourth polygon similar to $\mathcal{P}$. Summarizing we have:

Theorem 7.1 *Every simple polygon with n vertices, has a 4-disection with $2n-2$ pieces, $n \geq 3$.*

We now show how to obtain $(4+3k)$-dissections of $\mathcal{P}$ with at most $(2n-2) + k(2n + \lfloor \frac{n}{3} \rfloor - 6)$ pieces. Since the coloring of the vertices of T induces a partition on its vertices, there is a chromatic class with at most $\lfloor \frac{n}{3} \rfloor$ vertices. Suppose then that the cromatic class containing the vertices with color 1 has at most $\lfloor \frac{n}{3} \rfloor$ elements. Let $\mathcal{D}^{''}$ be the dissection of $\mathcal{P}$ obtained from $\mathcal{D}$ by adding cuts along the diagonals of T joining pairs if vertices colored with colors 2 and 3, see Figure 11(a) and (b). Observe that the number of diagonals of T with endpoints colored 2 and 3 is exactly the number of vertices of color 1 minus 1, and that when we cut the pieces of $\mathcal{D}$ along each of these diagonals, the number of pieces increases by 2, see

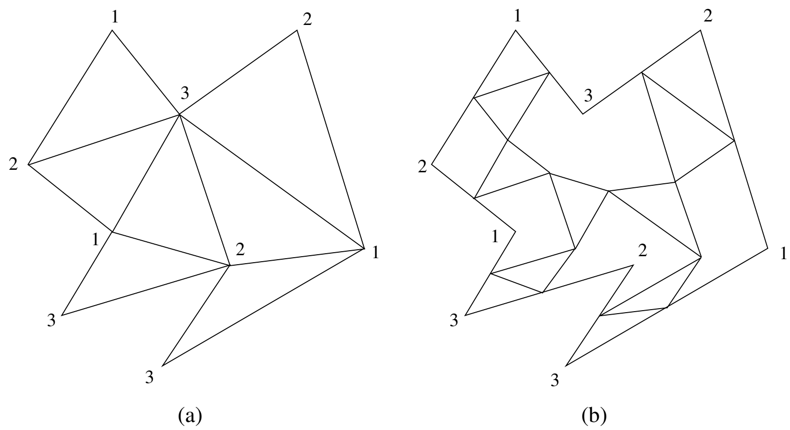

Figure 10: Coloring and obtaining our final dissection $\mathcal{D}$.

Figure 11. Since there are at most $\lfloor \frac{n}{3} \rfloor$ vertices with color one, we have that the number of pieces of $\mathcal{D}''$ is at most:

$$2n - 2 + 2(\left\lfloor \frac{n}{3} \right\rfloor - 1).$$

Observe now that the pieces of $S_1' = \{\mathcal{P}_i' : v_i \text{ has color } 1\}$ when assembled properly form a polygon $\mathcal{P}_1$ similar to $\mathcal{P}$, dissected along its diagonals corresponding to those of $\mathcal{P}$ joining pairs of vertices colored 2 and 3. Let $\mathcal{D}_1$ be the dissection induced in $\mathcal{P}_1$ by $\mathcal{D}''$. Combining $\mathcal{D}''$ with $\mathcal{D}_1$ we obtain a 7-dissection of $\mathcal{P}$ with at most

$$(2n-2) - \left\lfloor \frac{n}{3} \right\rfloor + 2n - 2 + 2(\left\lfloor \frac{n}{3} \right\rfloor - 1) = (2n-2) + 2n + \left\lfloor \frac{n}{3} \right\rfloor - 4$$

pieces. Clearly we can now iterate our previous procedure on the pieces of our last dissection of $\mathcal{P}$ containing the vertices of $\mathcal{P}$ with color 1 to obtain $4+3k$-dissections of $\mathcal{P}$ with $(2n-2) + k(2n + \lfloor \frac{n}{3} \rfloor - 4)$ pieces.

Thus we have proved:

Theorem 7.2 *Every simple polygon $\mathcal{P}$ with n vertices has a $4{+}3k$-dissection with $(2n-2) + k(2n + \lfloor \frac{n}{3} \rfloor - 4)$ pieces.*

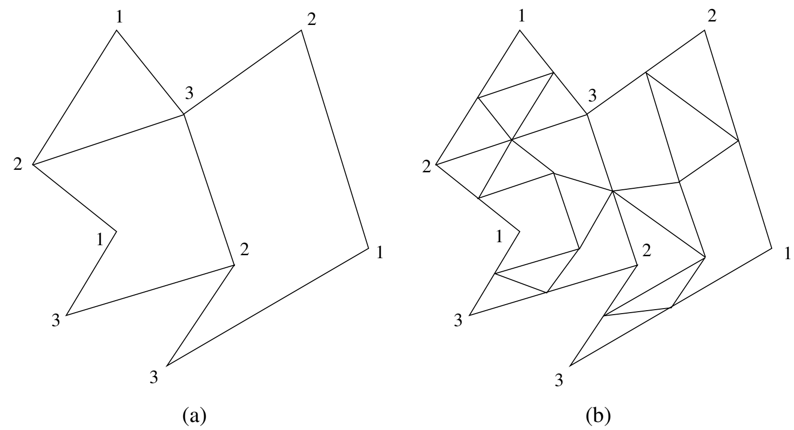

Figure 11: Obtaining $\mathcal{D}''$.

8 Star shaped polygons

A polygon $\mathcal{P}$ is called star shaped if there is a point p in $\mathcal{P}$ such that the line segment connecting it to any other point in $\mathcal{P}$ is contained in $\mathcal{P}$. We show how to obtain dissections of *star shaped* polygons with n vertices having $2kn+1$ pieces such that the piecesw of these disections can be reasembled to form 4, 7, ..., or $3k+1$ polygons similar to $\mathcal{P}$. We start by proving:

Theorem 8.1 *Any star shaped polygon $\mathcal{P}$ has a 4-sequential dissection $\mathcal{D}$ using $2n+1$ pieces, one of which is a star shaped polygon similar to $\mathcal{P}$.*

Let $\mathcal{P}$ be a star shaped polygon, and let p be a point in the interior of $\mathcal{P}$ such that the line segment connecting p to any point q in $\mathcal{P}$ is totally contained in $\mathcal{P}$. Suppose first that $\mathcal{P}$ has an even number of vertices. Color the vertices of $\mathcal{P}$ with colors 1 and 2 in such a way that adjacent colors receive different colors. Connect p to all the vertices of $\mathcal{P}$ to obtain a set of n triangles as shown in Figure 12(a). Subdivide the triangles obtained into 4 subtriangles using the mid-points of their edges as shown in the same figure. Next delete the edges connecting p to the vertices of $\mathcal{P}$, as in Figure 12(b) to obtain a dissection $\mathcal{D}$ of $\mathcal{P}$ with $2n+1$ pieces, one of which is similar to $\mathcal{P}$. Observe that all the pieces containing a vertex of color 1 (resp 2) can be regrouped to form a star shaped polygon similar to $\mathcal{P}$. The remaining n triangles can also be regrouped to form a fourth polygon similar to $\mathcal{P}$. The case when $\mathcal{P}$ has an odd number of vertices can be done in a similar way,

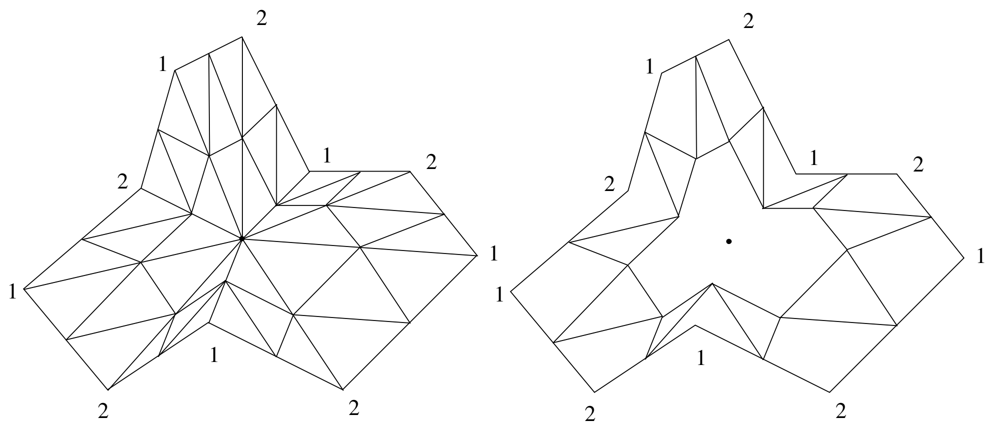

Figure 12: A 4-sequential dissection of a star shaped polygon wit n vertices, n even, using $2n+1$ pieces.

except that we color exactly one vertex of $\mathcal{P}$ with both colors 1 and 2. The details are left to the reader. An example for this case is shown in Figure 13.

By using the piece of $\mathcal{D}$ similar to $\mathcal{P}$, we can now easily obtain $3k+1$-dissections of $\mathcal{P}$ with $2kn+1$ pieces. Thus we have proved:

Theorem 8.2 *Any star shaped polygon with* n *vertices has* $3k+1$*-dissections with* $2kn+1$ *pieces.*

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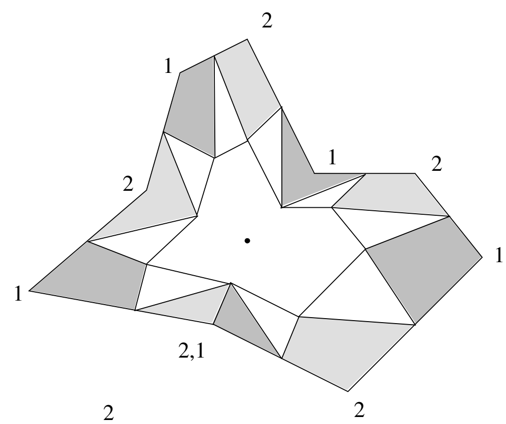

Figure 13: A 4-dissection of a star shaped polygon with n vertices, n odd, using $2n+1$ pieces.

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