MAXIMAL LENGTH COMMON NON-INTERSECTING PATHS

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Abstract

Given a set P_n of n points on the plane labeled with the integers $\{1, \ldots, n\}$, an increasing path of P_n is a sequence of points $i_1 < \ldots < i_k$ such that the polygonal path obtained by connecting i_j to i_{j+1}, $j = 1, \ldots, k-1$ is non-self intersecting. We show that any point set on the plane admits an increasing path of length at least $\sqrt{2n}$. We also study the problem of finding the longest common increasing path of two convex point sets on the plane and give an $O(n^2 \log n)$ time algorithm to find such a path.

1 Introduction

Let $P_n = \{p_1, \ldots, p_n\}$ be a set of n points on the plane. We say that P_n *supports* a planar graph $G(V, E)$ if there is a plane embedding of $G(V, E)$ on the plane in such a way that its vertices are mapped to the elements of P_n and its edges to straight line segments connecting pairs of adjacent vertices. Given two point sets P_n and Q_n, the problem of finding graphs supported by both of them has received attention recently. For example, Aronov, Seidel and Souvaine [1] and Kranakis and Urrutia [5] studied the problem of finding common triangulations of point sets of polygons; see also Shapira and Rappaport [7]. The problem of finding embeddings of trees on point sets has also been studied recently by Bose, McAllister and Snoeyink [2], Ibeke, Perles, Tamura and Tokunaga [4].

Let P_n and Q_n be labeled point sets, both labeled with the integers $1, \ldots, n$. A **non-intersecting path** π (henceforth called path) of P_n is a non-repeating sequence of points $i_1, \ldots, i_k$ such that the polygonal path obtained by connecting i_j to i_{j+1} , $j = 1, \ldots, k-1$ is non-self intersecting. In addition, we say that π is an *increasing* path of P_n if $i_1 < \ldots < i_k$.

In this paper, we study the problem of finding the longest common increasing path of P_n and Q_n, for the case when both P_n and Q_n are convex point sets, that is they are vertices of convex polygons. We give efficient algorithms to solve the longest common increasing path for this case. Our motivation arises also in part from the well known result of Erdös and Szekereres that states that any sequence of n numbers contains an increasing or decreasing subsequence of size $\sqrt{n}$. In particular, we prove that any labeled convex point set P_n always contains an increasing path of length $\sqrt{2n}$.

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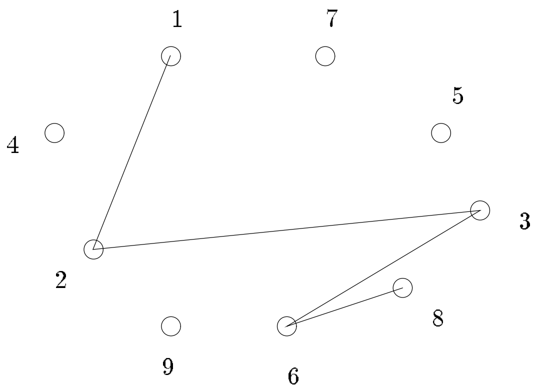

Figure 1: Point set P_n and its associated permutation $\sigma = (1, 7, 5, 3, 8, 6, 9, 2, 4)$. The increasing path obtained is $1, 2, 3, 6, 8$.

2 Longest increasing path

To start our paper, we show first that any convex point set P_n contains an increasing path of length $\sqrt{2n}$. This technique will then be extended to develop an algorithm to find the longest common path of the convex point sets.

Let π be the longest increasing path of P_n and suppose that π starts at point i and ends at a point k. If we read the labels of the elements of P_n in the clockwise order starting from index i, we obtain a permutation $\sigma(1), \sigma(2), \ldots, \sigma(n)$ of $\{1, \ldots, n\}$. See Figure 1.

The following observation will prove useful.

Observation 2.1 *The elements of π contained between i and k (resp. between k and i) in the clockwise order around P_n form an increasing (resp. decreasing) subsequence of π, ending at k.*

For example for the path shown in Figure 1, the elements $1, 3, 8$ form an increasing subsequence of $\{1, 7, 5, 3, 8, 6, 9, 2, 4\}$, and $8, 6, 2$ a decreasing subsequence.

We next observe that the elements of any increasing subsequence $\sigma^+(k)$ of σ ending at k, together with those of any decreasing subsequence $\sigma^-(k)$ starting at k, can be "merged" to obtain an increasing path of P_n. To achieve this, we first take the smallest element of the set $\sigma^+(k) \cup \sigma^-(k)$ and join it to the second smallest, etc. until we reach k. For example for the permutation in Figure 1, using the increasing and decreasing subsequences $\{1, 7, 8\}$ and $\{8, 6, 2\}$ ending and starting at 8 resp. we can obtain the increasing path $\{1, 2, 6, 7, 8\}$.

Next, we associate to every element i of σ a point on the plane with coordinates (x_i, y_i) such that x_i is the length of the longest increasing subsequence of σ ending at i, and y_i is the length of the longest decreasing subsequence of σ starting at i. For example the coordinates associated to point 6 in the same example are $(3, 2)$. It is well known that this mapping sends different elements of σ to different points on the plane; see for example Exercise 14.7 in [8]. Since the number of points on the plane with integer coordinates (i, j) such that $i + j < \sqrt{2n}$ is less than n, we conclude that there is a point k of σ that is mapped to a point (x_k, y_k) such that $x_k + y_k \geq \sqrt{2n}$. Thus we have proved:

Theorem 2.1 *Any convex point set P_n with n elements has an increasing path of length at least $\sqrt{2n}$.*

We notice that our method leads easily to an $O(n^2 \log n)$ time algorithm to compute the longest increasing path of P_n. To see this, we notice that for each i, $1 \leq i \leq n$, we can calculate (x_i, y_i) in $O(n \log n)$ time using well known algorithms to compute the longest increasing subsequence of a permutation [3]. This has to be repeated $O(n)$ times, yielding an $O(n^2 \log n)$ time algorithm. We do not believe that this is optimal, and conjecture the existence of an $O(n \log n)$ time algorithm to solve this problem.

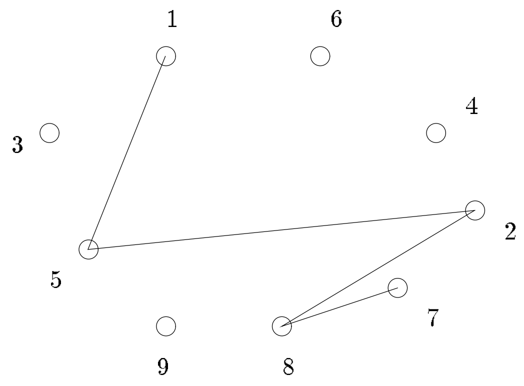

Figure 2: Point set Q_n and its associated permutation $\rho = (1, 6, 4, 2, 7, 8, 9, 5, 3)$. The path corresponding to the increasing path of P_n is $1, 5, 2, 8, 7$.

3 The longest common subsequence

In this section, we present an $O(n^2 \log n)$ time algorithm to solve the longest common increasing path of two convex sets P_n and Q_n. As in the previous section, let us assume first that we know that the initial point of π is i and assume that π ends at an unknown point k.

Using the same arguments as those used to prove Theorem 2.1, we can see that if we read the elements of P_n and Q_n in the clockwise direction starting from i, we get two permutations σ and ρ of $\{1, \dots, n\}$. In each of them, π defines common increasing and decreasing subsequence of σ and ρ ending at k. See Figure 2.

We now show how to find k in $O(n \log n)$ time. Given two permutations σ and ρ of $\{1, \dots, n\}$ and an element $i \in \{1, \dots, n\}$ we define $i(\sigma, \rho)$ (respectively $i'(\sigma, \rho)$) to be the length of the longest common increasing subsequence (resp. decreasing subsequence) of σ and ρ that ends at i (resp. starts at i). Thus finding k (and thus π itself) is equivalent to finding the point i of $\{1, \dots, n\}$ that maximizes $i(\sigma, \rho) + i'(\sigma, \rho)$. We now show how to find all $i(\sigma, \rho)$ in $O(n \log n)$ time.

Given σ and ρ, we define a *permutation diagram* of σ and ρ as follows: Consider two parallel line segments L_1 and L_2 each with n points labeled according to σ and ρ. Next join the point labeled i in L_1 to that labeled i in L_2 using a line segment l_i, $i = 1, \dots, n$. We now notice that any common increasing subsequence $j_1, \dots, j_s = i$ ending at i of σ and ρ corresponds to a set of mutually non-intersecting line segments $l_{j_1}, \dots, l_i$ contained to the left of l_i; see Figure 3. Using the permutation diagram of σ and ρ, we now obtain a new permutation γ of $\{1, \dots, n\}$ as follows: relabel the elements on L_1 in increasing order from left to right $1, \dots, n$. This induces a relabeling on the elements of L_2 as shown in Figure 3. Let γ be the permutation obtained by reading these labels on L_2 from left to right. Notice now that each common increasing subsequence of σ and ρ corresponds to exactly one increasing subsequence of γ. Again referring to the permutations of Figure 3, the increasing subsequence $1, 4, 6, 7$ of γ corresponds to the common increasing subsequence $1, 7, 8, 9$ of σ and ρ. But now using γ and results in [3], we can calculate all $i(\sigma, \rho)$ in $O(n \log n)$ time.

The permutation graph G obtained from the permutations σ and ρ is now defined as the graph with vertex set $\{1, \dots, n\}$ in which i and j are adjacent iff l_i intersects l_j. See Figure 3.

Using similar techniques, we can find for all i the length $i'(\sigma, \rho)$ of all common decreasing subsequences of σ and ρ that start i in $O(n \log n)$ time. Thus we have proved:

Lemma 3.1 *Given two permutations σ and ρ of $\{1, \dots, n\}$ we can find $i(\sigma, \rho)$, and $i'(\sigma, \rho)$, $i = 1, \dots, n$ in $O(n \log n)$ time.*

Consequently, we have:

Lemma 3.2 *Given two permutations σ and ρ of $\{1, \ldots, n\}$, we can find the path π as defined before in $O(n \log n)$ time.*

We can now prove:

Theorem 3.1 *Finding the longest common increasing path of two labeled convex point sets P_n and Q_n can be done in $O(n^2 \log n)$ time.*

Proof: For each $i \in \{1, \ldots, n\}$ find two permutations σ and ρ by reading the elements of P_n and Q_n in the clockwise direction starting at i. Find the longest common path π_i generated by these permutations as in Lemma 3.2 in $O(n \log n)$ time. Choose the largest π_i, and our result follows.

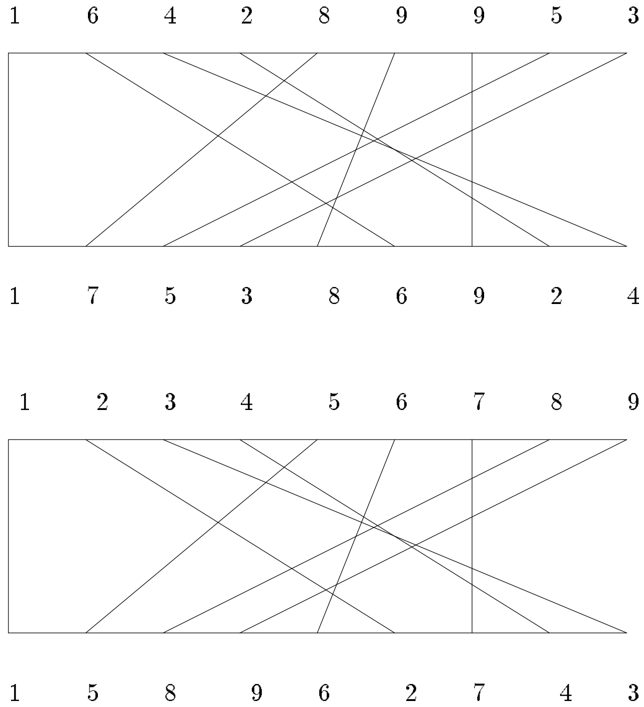

Figure 3: The top picture represents the permutation diagram of the permutations $\sigma = (1, 7, 5, 3, 8, 6, 9, 2, 4)$ and $\rho = (1, 6, 4, 2, 7, 8, 9, 5, 3)$, respectively. The bottom picture represents the permutation $\gamma = (1, 4, 8, 9, 6, 2, 7, 4, 3)$ obtained by relabeling the permutations σ and ρ.

4 Conclusions

We proved that any convex point set on the plane admits an increasing path of length $\sqrt{2n}$. We believe that this bound is not tight, and that the correct value is around $3\sqrt{n}$. An $O(n^2 \log n)$ algorithm was also obtained to find the longest increasing path of a convex point set. We believe that this algorithm is not optimal. An $O(n^2 \log n)$ algorithm to obtain the longest common increasing path of two labeled convex point sets was also obtained.

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