ISOMORPHIC TRIANGULATIONS WITH MINIMAL NUMBER OF STEINER POINTS

(Extended Abstract)

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Abstract

We present two algorithms for constructing isomorphic (i.e. adjacency preserving) triangulations of two simple n vertex polygons P, Q with k, l reflex vertices, respectively. The first algorithm computes an isomorphism by introducing at most $O((k+l)^2)$ Steiner points and has running time $O(n + (k+l)^2)$. The second algorithm computes an isomorphism by introducing at most $O(kl)$ Steiner points and has running time $O(n + kl \log n)$. The number of Steiner points introduced by the second algorithm is also optimal. This solves an open problem first proposed by Goodman and Pollack and subsequently elaborated by Aronov, Seidel and Souvaine (see [1]).

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1 Introduction

For many problems in image analysis it is necessary to transform one picture into another [5]. This is often accomplished by introducing Steiner points in order to achieve isomorphic triangulations. Let P, Q be two n vertex polygons with vertices $v_1, \ldots, v_n$ and $u_1, \ldots, u_n$, respectively. The (triangulated) polygons are called isomorphic if there exists a one-to-one correspondence ϕ of $\{1, \ldots, n\}$ into itself such that $\{v_i, v_j\}$ is an edge of P if and only if $\{u_{\phi(i)}, u_{\phi(j)}\}$ is an edge of Q. In general, such an isomorphism may not exist (see [1]). To resolve this problem it is necessary to introduce Steiner points at approriate locations. This raises the following question: "Give an efficient algorithm that constructs isomorphic triangulations of the polygons P, Q by introducing the minimum possible number of Steiner points."

A related question was first proposed by Goodman and Pollack in 1989 (see [1]). The first solution was suggested by Aronov, Seidel and Souvaine in [1]. They give an algorithm for finding isomorphic triangulations of two simple polygons that requires the introduction of $O(n^2)$ Steiner points. The running time of their algorithm is $O(n^2)$.

In this paper we present two algorithms for constructing isomorhic triangulations between two simple polygons P, Q with the same number of vertices. The first algorithm computes an isomorphism by introducing at most $O((k+l)^2)$ Steiner points and has running time $O(n+(k+l)^2)$. Steiner points are introduced via the technique of "optimal paths". The second algorithm computes an isomorphism by introducing at most $O(kl)$ Steiner points and has running time $O(n + kl \log n)$. In this case, Steiner points are introduced via the technique of "ray-shooting".

In the next two sections we outline the algorithms and proofs of correctness; details will appear in the full paper.

2 Isomorphism via Optimal Paths

To make things easier, let us assume that we have two n vertex polygons P and Q with vertices $v_1, \ldots, v_n$ and $u_1, \ldots u_n$, respectively, and that we want to find isomorphic triangulations of P and Q in which v_i is mapped to u_i, for $i = 1, ..., n$.

Theorem 2.1 *Given two polygons P and Q with k and l reflex vertices respectively, we can find isomorphic triangulations for them in $O(n+(k+l)^2)$ time by introducing $O((k+l)^2)$ Steiner points.*

PROOF (OUTLINE) Assume that P has k reflex vertices $v_{s_1}, \ldots, v_{s_k}$ and Q has l reflex vertices $u_{t_1}, \ldots, u_{t_l}$. Our objective is to show that P and Q have isomorphic triangulations after introducing $O((k+l)^2)$ Steiner points.

Define the set $Z = \{s_1, \ldots, s_k\} \cup \{t_1, \ldots, t_l\}$ and suppose that the elements of Z are $z_1, \ldots, z_m$ in sorted order where m is at most $k+l$. Z contains the indicess of the reflex vertices of P and Q (see Figure 1).

We now define two simple polygons P' and Q' as follows:

1. Polygon P:
   (a) For each vertex v_{z_i} of P, compute the shortest polygonal path $P(i)$ path from v_{z_i} to $v_{z_{i+1}}$, for $i = 1, ..., m$, where $m+1 = 1$. Let P'' (not P') be the concatenation of all $P(i)$, for $i = 1, ..., n$. (see Figure 2).
   (b) We next observe that P'' has m vertices (although some of them appear twice on P''). This follows from the observation that by the definition of P'' the vertices of P'' are of the form v_{z_i} or reflex vertices of P appearing on a shortest path from some v_{z_i} to $v_{z_{i+1}}$, which in any case, are already accounted for in Z.
      i. Defining P':
      If it so happens that P'' is simple then take $P' = P''$, otherwise:
      ii. If P'' is not simple, notice that by splitting some of its vertices into two (these vertices are reflex vertices of P), we can obtain a simple polygon P' totally contained in P. Clearly the number of vertices of P' is at most $m+k$ (see Figure 3).
2. Polygon Q:
   (a) We define Q' in a similar way but, using Q instead of P.

Map v_{z_i} on P' to u_{z_i} on Q' and extend this mapping in the natural way to all duplicate vertices introduced in (b.ii) introducing extra vertices as necessary on the edges of P' and Q' so as to ensure that every duplicate vertex of P' (or Q') maps to a different vertex of Q' (P'). It is not hard to see that in this last step, the number of vertices of P' and Q' increases to at most $m + k + l \leq 2(k+l)$.

It now follows that P' and Q' have isomorphic triangulations with $O((k+l)^2)$ Steiner points.

Now $P \backslash P'$ and $Q \backslash Q'$ can be decomposed into m simple (spiral) polygons $P_1, \ldots, P_m$ whose boundaries can be decomposed into a convex and a reflex chain. The convex chain is that joining v_{z_i} to $v_{z_{i+1}}$ on P (resp. u_{z_i} to $u_{z_{i+1}}$ on Q) and the reflex chain is the shortest path from v_{z_i} to $v_{z_{i+1}}$ in P (resp. u_{z_i} to $u_{z_{i+1}}$ in Q).

Now we can prove the following lemma.

Lemma 2.1 *Let P and Q be spiral polygons with vertices $v_1, \ldots, v_n$ and $u_1, \ldots, u_n$ such that $v_1, \ldots, v_k$ and $u_1, \ldots, u_k$ are all the reflex vertices of P and Q resp. Then there are isomorphic triangulations of P and Q in which v_i is mapped to u_i ,$i = 1, ..., n$ containig at most $O(k^2)$ Steiner points.*

Let r_i be the number of reflex vertices of P_i, then by the previous paragraph, there are isomorphic triangulations between P_i and Q_i using $O(r_i^2)$ Steiner points. However since the sum $r_i + \cdots + r_m$ is $O(k + l)$ it follows easily that

$$r_i^2 + \cdots + r_m^2 = O((k + l)^2)$$

and our result now follows.

Complexity.

The only part that has to be analyzed here is the construction of P'' and Q''. Next we will show how to do this in linear time.

First obtain a triangulation H of P using Chazelle's linear time algorithm [2], and consider the duat tree T of H. Observe that P'' intersects each edge in H at most twice. It is well known that the shortest path between any two points of P corresponds to a path in T. Moreover by the previous observation each edge of T is used at most twice. Now we can prove that

Lemma 2.2 *Calculating P'' (and Q'') can be done in linear time.*

PROOF (OUTLINE) Mark on T those vertices corresponding to triangles containing vertices v_{z_i}. Delete from T by standard leaf prunning techniques all those triangles not intersected by P''. We can now calculate in linear time the paths on T from v_{z_i} to $v_{z_{i+1}}$ and thus the shortest paths from v_{z_i} to $v_{z_{i+1}}$ in total linear amortized time. ∎

This completes the outline of the proof of the theorem. ∎

3 Isomorphism via Ray-Shooting

The previous algorithm needs to introduce $O((k + l)^2)$ Steiner points. Thus the number of Steiner points is proportional to the square of the largest number of reflex vertices among the given polygons. In the sequel we give an algorithm that uses only $O(kl)$ Steiner points but at the cost of a $\log n$ time-overhead.

Theorem 3.1 *Given two polygons P and Q with k and l reflex vertices respectively, we can find isomorphic triangulations for them in $O(n+kl\log n)$ time by introducing $O(kl)$ Steiner points.*

Proof (Outline) The idea is to establish the desired triangulation of the polygon P on the vertices of a convex polygon (say, a circle). The introduction of Steiner points corresponds to a planar partition of the circle. Now we overlap the planar partition of P with the planar partition of Q. From this we can deduce easily the equivalent triangulation of the polygons. Details are as follows.

1. Polygon P:
   (a) Map the n vertices of P on n equidistant verices all lying on a circle.
   (b) Move clockwise along the vertices and recursively eliminate all reflex vertices of the polygon P. To eliminate a reflex vertex, extend the two edges of the reflex vertex inside the polygon and shoot a ray within the region delimited by these two extended edges (see Figures 5 and 4) until it intersects an edge of the polygon, say the edge determined by the vertices v_i, v_{i+1}. Introduce this point of intersection between v_i and v_{i+1} as a Steiner point. Now map this Steiner point between the images of these two vertices on the circle. In addition, form a planar subdivision that updates the polygon by incorporating the new edge from the reflex vertex to the Steiner point introduced.
   (c) This introduction of Steiner points partitions the polygon into $k+1$ convex regions. On the circle this corresponds to a planar subdivision of the circle into $k+1$ corresponding convex regions. In view of the equivalent numbering it also follows that any triangulation of a region in the polygon has an isomorphic triangulation on the corresponding region on the circle.
2. Polygon Q: (see Figure 6)

(a) In essence we execute the same algorithm as above with Q replacing P but we use the following "overlapping" procedure. We keep track of new Steiner points introduced via a standard planar subdivion data-structure [4]. We traverse the reflex vertices of Q one at a time moving clockwise. For each such vertex we introduce a Steiner point on an edge of the polygon Q (as in the case of the polygon P); in addition we introduce as Steiner points the points of intersection of the ray and the edges of the planar subdivision corresponding to P. After introducing these Steiner points we update the data structure and proceed to the next reflex vertex of Q.

Complexity.
We observe that

Lemma 3.1 *The above ray-shooting divides the polygon into subpolygons such that the boundary of each subpolygon can be broken into two chains one convex and the other not necessarily convex but totally contained in the boundary of the original polygon.*

Using this and standard ray-shooting algorithms [3] it is easy to see that the complexity of introducing a Steiner point is $O(\log n)$ per Steiner point. Hence the total cost is $O(kl \log n)$.

An important feature of the algorithm is that it requires a date structure for maintaining and introducing Steiner points. Such a data structure is the planar subdivision [4]. The main cost is in the "overlapping" procedure of the two circular planar graphs. Planarity implies that each internal edge of the Q intersects at most k internal edges of P. This determines a total of at most $O(kl)$ points of intersection. These new points are introduced as the new Steiner points.

The resulting regions can now be triangulated thus resulting in equivalent triangulations for the given simple polygons. This completes the proof of the theorem. ■

4 Comparison of the two algorithms

It easy to see that the first algorithm is superior to the second only when the polygons have the same number of reflex verices up to an $O(\log n)$ factor.

More precisely this is true exactly when

$$\frac{k}{l} + \frac{k}{l} = O(\log n).$$

5 Lower Bounds

The number of Steiner points introduced by the second algorithm is optimal up to a constant factor. This follows from the following result.

Theorem 5.1 *There exist two simple polygons P, Q any isomorphic triangulation of which requires at least $\Omega(kl)$ Steiner points.*

PROOF (OUTLINE) The example given in [1] is sufficient to prove the lower bound. ■

6 Conclusion

We have given two algorithms for computing isomorphic triangulations of any two simple polygons with the same number n of vertices. We related the number of Steiner points introduced with the number of reflex vertices, say k, l of the two polygons. The second algorithm is optimal in the number of Steiner points intorduced (thus answering a question of [1]), while the first algorithm has superior running time when $\frac{k}{l} + \frac{k}{l} = O(\log n)$. The question of determining the least number of Steiner points needed still remains open [1].

Acknowledgements

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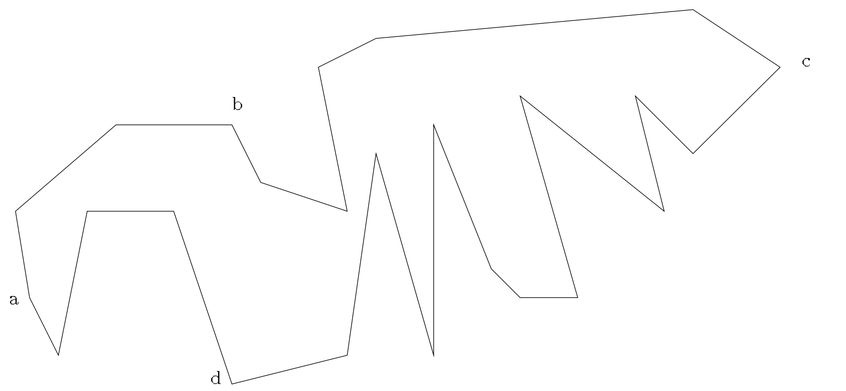

Figure 1: Polygon P; a, b, c, d are the reflex vertices of Q.

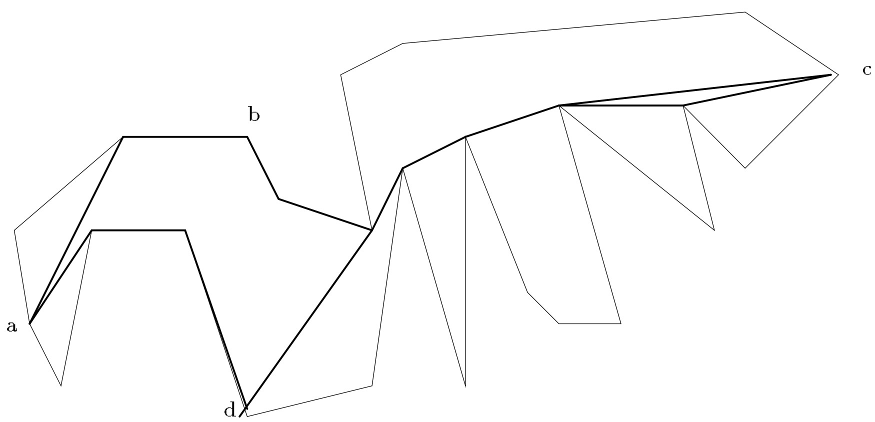

Figure 2: Computing shortest paths.

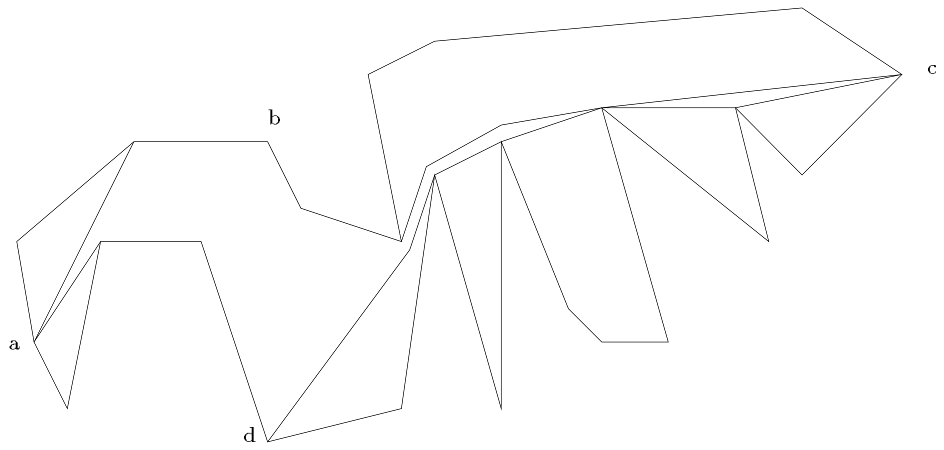

Figure 3: Splitting some vertices of the shortest path.

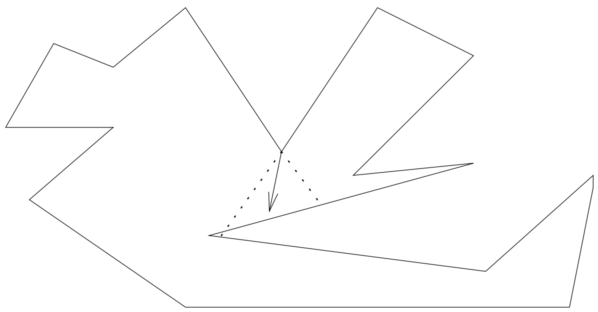

Figure 4: Introducing Steiner points via ray-shooting

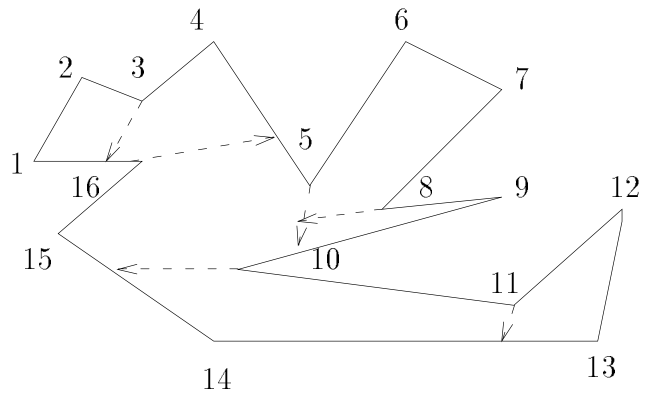

Figure 5: Polygon P: Introducing Steiner points via ray-shooting

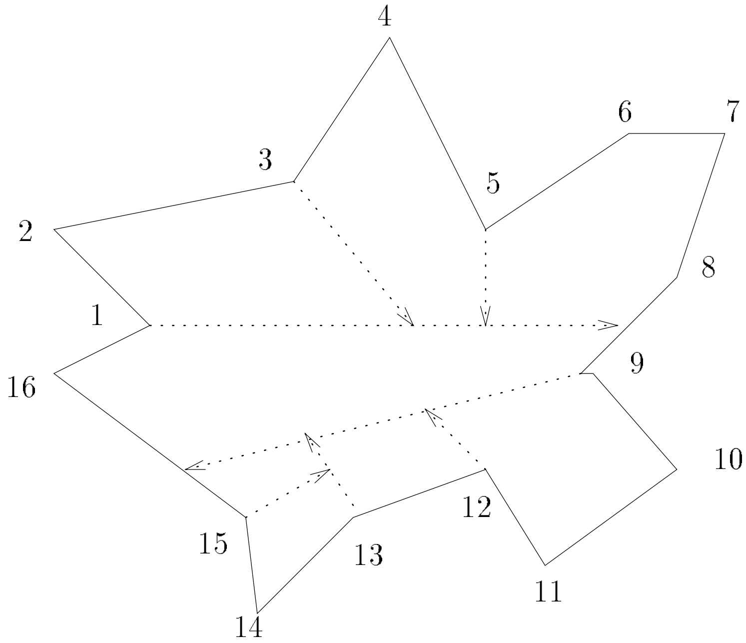

Figure 6: Polygon Q: Introducing Steiner points via ray-shooting

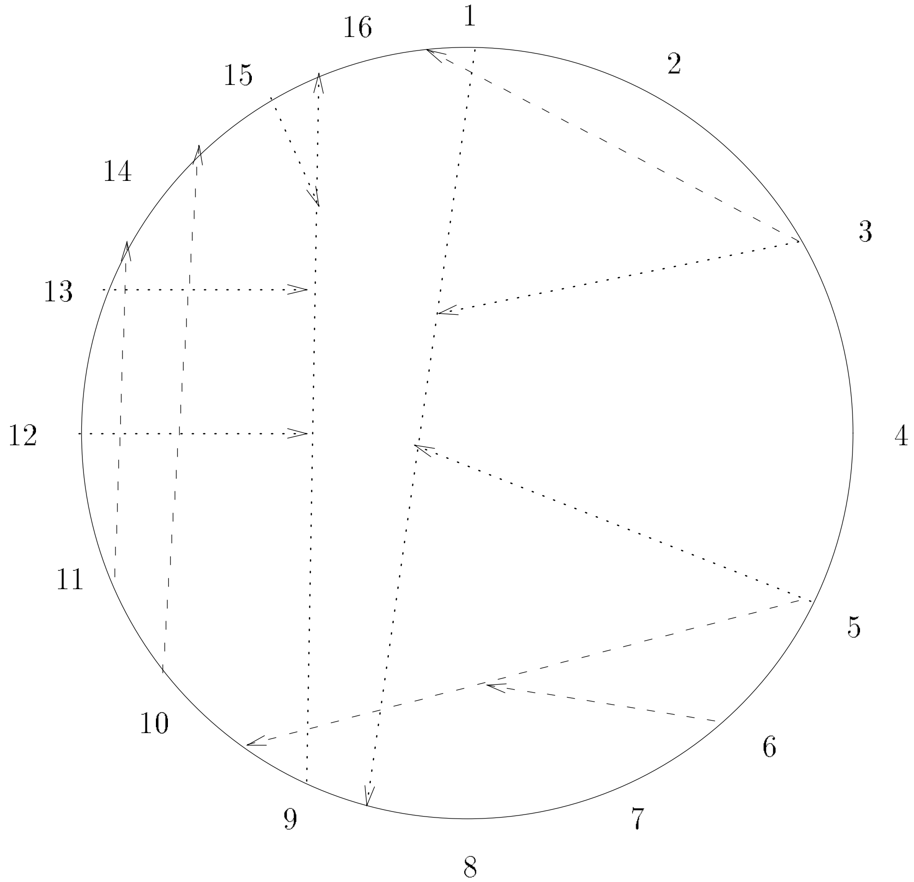

Figure 7: Mapping P and Q onto a convex polygon

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