THE FLOODLIGHT PROBLEM*

Prosenjit Bose[1] **Leonidas Guibas**[2] **Anna Lubiw**[3]

Mark Overmars[4] **Diane Souvaine**[5] **Jorge Urrutia**[6]

Abstract

Given three angles summing to 2π, given n points in the plane and a tripartition $k_1 + k_2 + k_3 = n$, we can tripartition the plane into three wedges of the given angles so that the i-th wedge contains k_i of the points. This new result on dissecting point sets is used to prove that lights of specified angles not exceeding π can be placed at n fixed points in the plane to illuminate the entire plane if and only if the angles sum to at least 2π. We give $O(n \log n)$ algorithms for both these problems.

1. Introduction

Illumination problems have been a source of many interesting results in computational geometry, for example in the area of Art Gallery theorems and algorithms—see [O'R], [S]. The usual scenario is that we have some target objects in two or more dimensions that are to be illuminated, and some specified sites for lights, which are assumed to shine light in every direction—i.e. with an angle of illumination of 360° in the planar case. See [CRU] and [CRCU] for some recent results of this nature.

In this paper we will consider a variant of these problems in which lights are constrained to shine in some specified angles of illumination: Given n points in the plane which are to be the positions of n floodlights, and given n planar angles representing the arcs of illumination of the floodlights, decide how to assign the floodlights to the points and how to fix their rotational angles, in order to light up some target. A harder problem is to minimize the number of floodlights needed. We will consider two types of target: a line segment (or "stage"), and the whole plane.

At a recent workshop, Jorge Urrutia posed the version of this problem for lighting up a stage. This "Stage Light" problem seems difficult. In section 3 we give a counterexample to an intuitively plausible greedy algorithm.

---

* Part of this work was carried out when the authors were participants of the 1992 Workshop on Graph Theory and Computational Geometry at the Bellairs Research Institute of McGill University. A preliminary version appeared in the Proceedings of the 5th Canadian Conference on Computational Geometry, 1993.

1 Dept. Computer Science, McGill Univ., Montreal, Canada. jit@muff.cs.mcgill.ca

2 Computer Science Dept., Stanford Univ., Palo Alto, California, USA 94301. guibas@src.dec.com

3 Dept. Computer Science, Univ. of Waterloo, Waterloo, Canada, N2L 3G1. Supported in part by NSERC. alubiw@uwaterloo.ca

4 Dept. Computer Science, Univ. of Utrecht, P.O. Box 80.089, 3508 TB Utrecht, The Netherlands. Supported in part by the Netherlands Organization for Scientific Research (NWO). markov@cs.ruu.nl

5 Dept. Computer Science, Rutgers Univ., New Brunswick, New Jersey, USA 08903. Supported in part by NSF Grant CCR-91-04732. dls@cs.rutgers.edu

6 Dept. of Computer Science, Univ. of Ottawa, Ottawa, Ontario, Canada, K1N 6N5. jorge@csi.uottawa.ca

Section 2 concerns the problem of lighting up the whole plane. In the case that all the angles are bounded by π, we give a simple necessary and sufficient condition for lighting the plane: the given angles must sum to at least 2π. In the course of proving this we give a result on tripartitioning the plane: Given three angles $\theta_1, \theta_2, \theta_3$, with sum 2π, given n points in the plane, and a tripartition $k_1+k_2+k_3 = n$, the plane can always be partitioned into three wedges W_1, W_2, W_3 such that W_i has angle θ_i and contains k_i of the given points. The special case where the θ_i's are equal was proved earlier in [TN1], generalized to higher dimensions. Our tripartition result fits into a large family of results on dissections of point sets, of which the most famous is the Ham Sandwich Theorem (see [E, Chapter 4]). We give $O(n \log n)$ algorithms for tripartitioning, and for the floodlight problem. Our model of computation is the real RAM (see [PS]).

Other Work

Tokuyama and Nakano [TN1] proved the special case of the tripartitioning result when the three given angles are equal. Their interest was in solving the minimum weight one-to-many matching problem in a complete bipartite graph. They showed that this problem is equivalent to the tripartitioning problem in dimension t, with t equal cones. Such equal cones arise from the faces of a regular simplex centered about the origin. Tokuyama and Nakano proved that tripartitioning is always possible in this case, and gave an efficient randomized algorithm. In [TN2] they generalized to weighted points.

A number of results have been obtained since our initial work. Czyzowicz, Rivera-Campo, and Urrutia [CRCU2] gave a nice and efficient algorithm for a variant of the Stage Light problem in which one may choose the angles of the lights—again placing the lights at fixed points to illuminate a line segment—with the goal of minimizing the total sum of the angles used. Steiger and Streinu [private communication, 1993] have given a linear time algorithm for the tripartitioning problem, and a lower bound of $\Omega(n \log n)$ for the floodlight problem, in the case where no angle is greater than π. They have also shown that the general floodlight problem—with angles possibly greater than π—is in NP. Rote [private communication, 1993] has devised an alternate proof of our floodlight theorem which he can generalize to 3 dimensions in the case that the cones of the lights arise from a polytope enclosing the origin, where each cone is determined by the origin as apex, and by one facet of the polytope. He has a counterexample to a more general 3-dimensional result.

A related result in d dimensions is due to Pach and Rogers [private communication, 1992]: for any polytope P in d dimensions, the *dual cones* to the solid angles of the convex hull vertices cover the whole space. Each vertex v of P has a solid angle that can be extended to a full cone C_v; the *dual cone* of C_v consists of all rays starting at v that determine acute angles with every ray belonging to C_v.

Definitions

A *wedge* is the closed area of the plane bounded by two rays emanating from a common point.

Its *complementary wedge* is the wedge of the same angle formed by rotating each of the bounding rays by 180°. If the angle of a wedge is greater than π, then the wedge and its complement intersect in a region of positive area.

2. Illuminating the Plane

In this section we will give a simple necessary and sufficient condition for placing n small lamps at n points to light up the whole plane. By a "small" lamp we mean one that shines light in a wedge of no more than π radians. In this problem the n points are specified, and the angles of the n lamps are specified. We have the freedom to assign the lamps to the points, and then to rotate each lamp about its point.

Theorem 2.1. Given n points in the plane, and n angles $\alpha_1, \ldots, \alpha_n$, where each α_i is at most π, lights of the given angles can be placed at the given points to light up the whole plane if and only if $\sum_{i=1}^{n} \alpha_i \geq 2\pi$. Furthermore, in case the angles do add up to at least 2π, there is an $O(n \log n)$ time algorithm to place the lights at the points to illuminate the plane.

The proof of the theorem will show that we can choose the ordering of the α_i's as they appear from a "circle at infinity". To make this precise, suppose that lights of the given angles have been placed at the points to light up the whole plane. Consider a circle containing all the given points, and containing any intersection point of two of the bounding rays of the positioned lights. Each light illuminates some arc of this circle, and the lights can be cyclically ordered by the clockwise order of the first endpoint of the corresponding arcs on the circle. This ordering is independent of the choice of the circle. It is in this sense that placing the lights at the points determines a cyclic ordering of the lights. The proof of the theorem will show that any cyclic ordering of lights can be realized.

Theorem 2.1 is not true in general if one of the given angles is greater than π: for example the three angles $330°, 15°, 15°$, and the three points of an equilateral triangle cannot be used to light the plane.

The optimization version of the problem—to minimize the number of floodlights required to light the plane—can be solved in the case where all the angles are bounded by π: simply discard as many small angles as possible while maintaining a sum of at least 2π.

Our proof of Theorem 2.1 will provide an efficient algorithm to place the lights. We will divide the plane up into wedges, and light the wedges separately. The following lemma gives a sufficient condition for lighting a wedge.

Lemma 2.2. Let W be a wedge of angle $\theta \leq \pi$, let P be a set of $k \geq 1$ points in the complementary wedge, and let $\alpha_1, \ldots, \alpha_k$ be angles with $\sum \alpha_i \geq \theta$. Then lights of the given angles can be placed at the given points to light up the whole wedge W. Furthermore, there is an algorithm to position the lights that runs in time $O(k \log k)$.

The condition given in the lemma is sufficient for lighting a wedge, but not necessary. The lemma is not true in general for $\theta > \pi$: for $k = 1$, a point in the interior of the wedge and its complement cannot be used to light the whole wedge.

Proof of Lemma 2.2. Suppose that W consists of the rays r and s, emanating from the point x, together with the counter-clockwise angle θ from r to s. See Figure 2.1.

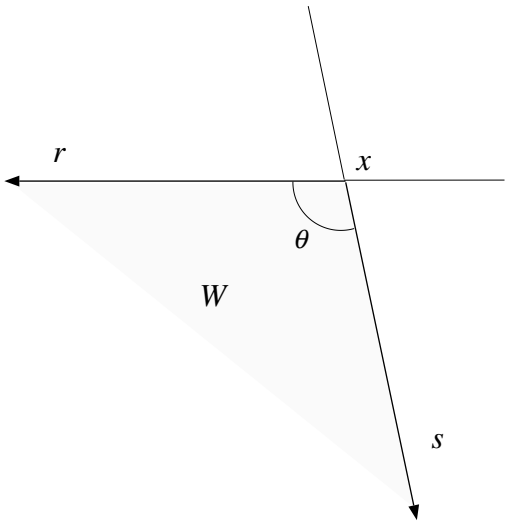

Figure 2.1

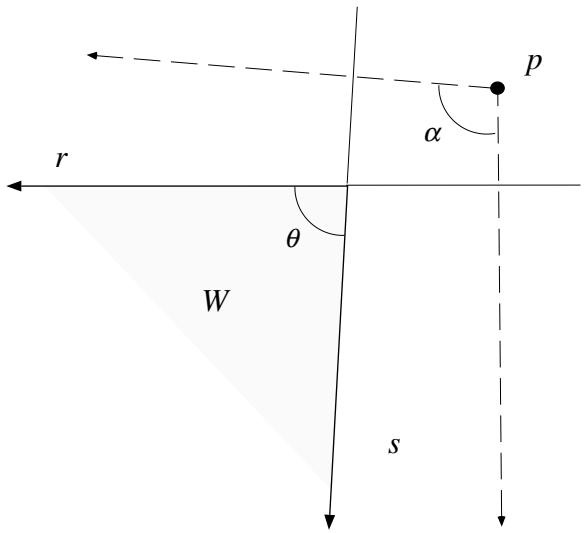

Figure 2.2

The proof will be by induction on k. In the general step we will reduce to problems of roughly half the size in order to obtain an efficient algorithm. If $k = 1$ then, since $\alpha_1 \geq \theta$ and the single point of P is in the complementary wedge, we can place a light of angle $\alpha = \alpha_1$ at the point to light the whole wedge. See Figure 2.2.

For $k > 1$, let $k' = \lfloor \frac{k}{2} \rfloor$ and let $\alpha' = \alpha_1 + \alpha_2 + \cdots \alpha_{k'}$ and $\alpha'' = \alpha_{k'+1} + \cdots + \alpha_k$. If $\alpha' \geq \theta$ then we can simply throw away the other angles and half the points. So assume $\alpha' < \theta$. Consider the family L of directed parallel lines making a counter-clockwise angle of $\theta - \alpha'$ from the line of ray r. There is some member of L having k' points of P to its right, and $k - k'$ points of P to its left. Let l be such a line. Let W' be the wedge of angle α' formed by the directed line l and the directed line of s. Let W'' be the wedge of angle $\theta - \alpha'$ formed by the directed line l and the directed line of r. See Figure 2.3. Observe that W' and W'' together cover W.

The complement of W' contains k' points, so by induction we can place lights of angles $\alpha_1, \alpha_2, \ldots, \alpha_{k'}$ at these points to light the wedge. The complement of W'' contains $k - k'$ points so by induction we can place lights of angles $\alpha_{k'+1}, \ldots \alpha_k$ at these points to light the wedge. (Note that the angle of the wedge, $\theta - \alpha'$, is less than or equal to the sum of the angles of the lights, α''.)

Note that this proof is algorithmic: we can find the line l in $O(k)$ time using median finding

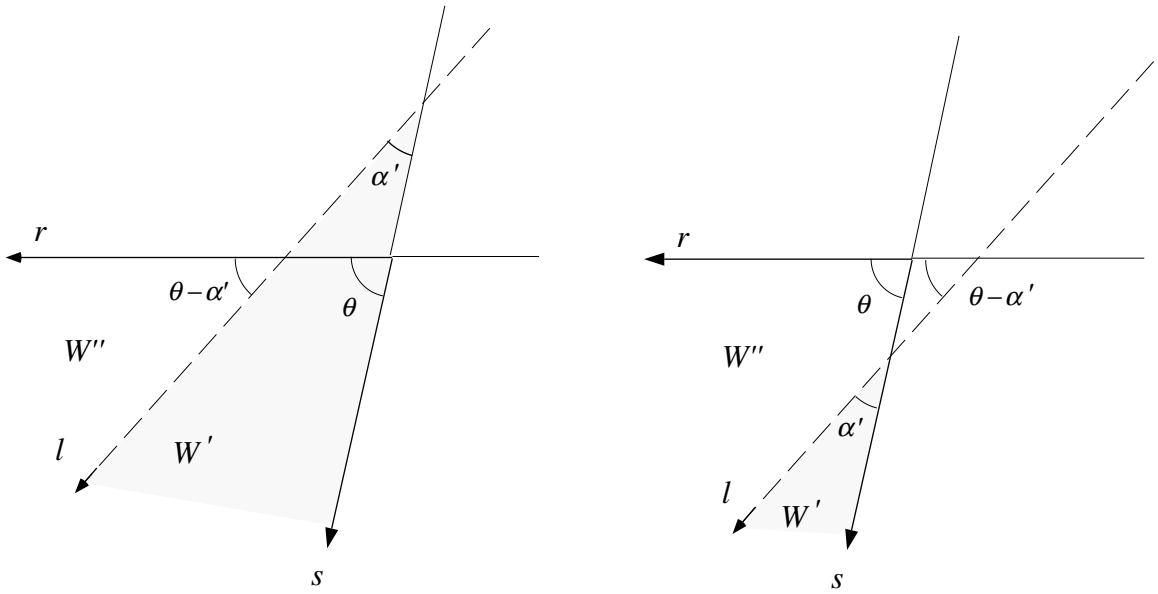

Figure 2.3

[BFP], and then we have decomposed the problem into two subproblems of half the size. Thus the total time required is $O(k \log k)$.

Finally, note that the ordering of the α_i's—upon which we based the partition into the two subproblems—was arbitrary, and that, by construction, the ordering of the α_i's about a circle at infinity will match the initial ordering. ∎

This lemma immediately implies one case of Theorem 2.1: suppose that the angles α_i can be partitioned into two sets A_1, of size k, and A_2, of size $n-k$, such that the sum of the angles in each set is π. (Of course, it is a difficult problem to *determine* if the angles can be partitioned in this way—we are claiming conceptual simplicity for this special case, not algorithmic simplicity.) Find a line that has k of the given points to one side and $n-k$ points to the other side. Let H_1 and H_2 be the half-planes formed by the line, where H_1 contains k points and H_2 contains $n-k$ points. We will apply Lemma 2.2 twice: once to the wedge that is the half-plane H_1, the $n-k$ points in the complementary half-plane, and the angles of A_2; and once to the wedge that is the half-plane H_2, the k points in the complementary half-plane, and the angles A_1. Since the hypotheses of Lemma 2.2 are satisfied, we can light the two wedges, and thus the whole plane.

In general the angles will not partition in two so neatly—and in any case, we cannot test whether they do—so we will instead partition the angles into three sets, such that the sum of the angles in each set is at most π. (This is always possible, as will be shown in the proof of Theorem 2.1.) In order to apply Lemma 2.2 we must tri-partition the plane into three wedges with the appropriate angles and the appropriate number of points in each wedge. This is a natural generalization of bipartitioning the plane into two half-planes with a specified number of points in

each half.

Theorem 2.3. Let θ_1, θ_2, and θ_3 be three non-trivial angles summing to 2π. Let P be a set of n points in the plane, and let $k_1 + k_2 + k_3 = n$ be a partition of n. Then there is a point x in the plane and three disjoint wedges W_1, W_2, W_3 emanating from x such that for each $i = 1, 2, 3$, wedge W_i has angle θ_i and contains k_i points of P. Furthermore, there is an algorithm to find x and the W_i's in time $O(n \log n)$.

Observe that we do not restrict the θ_i's to be less than π. Note that the theorem includes the case of bipartitioning, by uniting two of the wedges into one. Theorem 2.3 has the same flavour as many other results on dissecting point sets—see Chapter 4 of Edelsbrunner's book [E]. As for many of those results, a continuous version is also true: we can replace the discrete point set by a convex body whose area must be tripartitioned into specified portions by specified angles.

Proof of Theorem 2.3. At most one of the angles can be greater than or equal to π, so suppose that θ_1 and θ_2 are less than π. Suppose the wedges are ordered clockwise. Let r_1 be the ray between W_1 and W_2, r_2 be the ray between W_2 and W_3, and r_3 be the ray between W_3 and W_1. See Figures 2.4 and 2.5. Choose an initial orientation of the rays r_1, r_2, r_3 so that no line parallel to one of the rays goes through two or more of the given points. This orientation will be fixed from now on, and we will simply translate the configuration in the plane to achieve our goal.

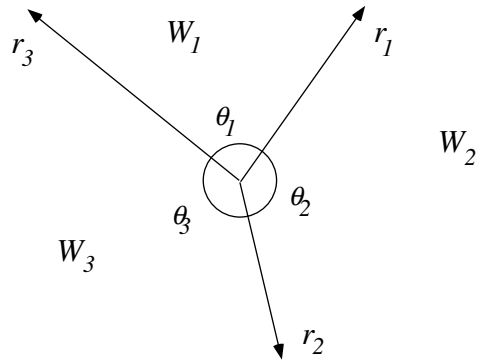

Figure 2.4

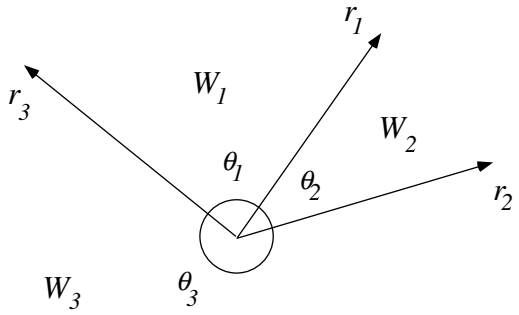

Figure 2.5

Among the family of directed lines parallel to r_1, consider which one is appropriate for r_1. We must have at least k_2 points to the right of the line because we need that many points in W_2. Similarly we must have at least k_1 points to the left of the line because we need that many points in W_1. Given a line l_1 that meets these condition, we must place r_2 so that it makes a clockwise angle θ_2 from l_1 and so that k_2 points lie above it in W_2. Let x_2 be a point on line l_1 from which r_2 can emanate. Similarly, let x_3 be a point on line l_1 from which r_3 can emanate so that it makes a counter-clockwise angle θ_1 from l_1 and has k_1 points above it in W_1. See Figure 2.6. If x_2 and x_3 coincide then we have the desired configuration. When l_1 is at its rightmost position, there are

exactly k_2 points to the right of it, and x_2 can be as far in the negative direction along l_1 as we wish—certainly below x_3. On the other hand, when l_1 is at its leftmost position there are exactly k_1 points to its left, and x_3 can be below x_1. Thus, by continuity, there must be some intermediate position for l_1 where x_2 and x_3 coincide.

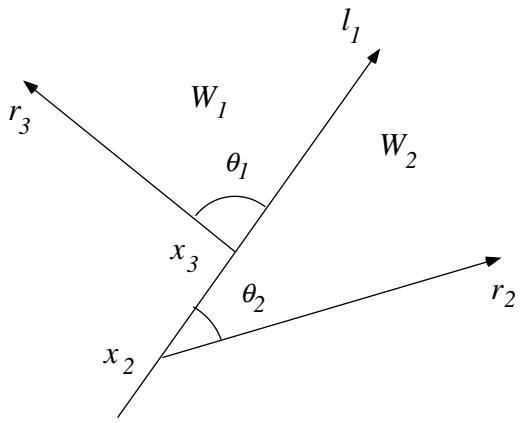

Figure 2.6

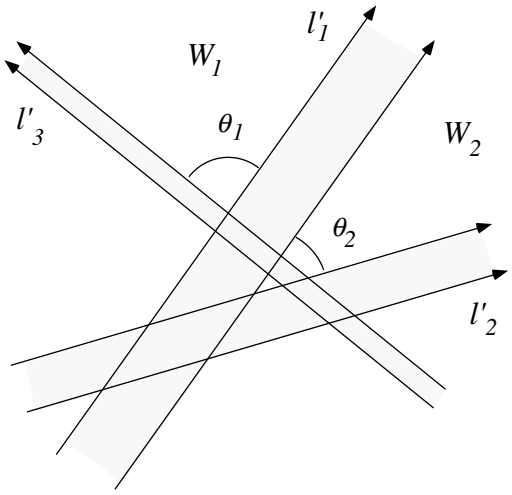

Figure 2.7

This existence proof can be turned into an $O(n \log n)$ algorithm. We will use binary search to find the line of r_1. After choosing the initial orientation for r_1, r_2, r_3, sort the points in the direction perpendicular to r_1. We cannot do a discrete search, fixing the line of r_1 at one of the n points, because it is possible that the unique solution has the line of r_1 lying somewhere strictly between two points in this ordering. Thus at each step of the binary search we will choose a *pair* of consecutive points in the ordering, and test the possibility of locating the line of r_1 somewhere in the range l'_1 between these two points. In case of failure we must know whether to move the range l'_1 left or right.

Given such a trial range l'_1 partition the points into R and L, the set of points to the right and left, respectively, of l'_1. We must position r_2 so that k_2 points of R lie in W_2. This constraint gives a range l'_2 of possible lines for r_2. We can find l'_2 using a linear time selection algorithm [BFP]. Similarly, we must position r_3 so that k_1 points of L lie in W_1, and we obtain a range l'_3 of possible lines for r_3. See Figure 2.7. If these three ranges l'_1, l'_2, l'_3 have a common intersection point x then the desired solution is obtained when the wedges emanate from that point. Otherwise the intersection of l'_2 and l'_3 lies either to the left or to the right of l'_1. As justified by the above proof, l'_1 must be moved in the opposite direction. Thus each step of our binary search takes $O(n)$ time, and we can find the desired configuration in $O(n \log n)$ time. ∎

Theorem 2.3 does not extend to 4-partitions: it is not possible to partition the plane into

wedges of angles $\theta_1 = 15°, \theta_2 = 165°, \theta_3 = 15°, \theta_4 = 165°$ counter-clockwise in that order so that they contain $k_1 = 2, k_2 = 0, k_3 = 1, k_4 = 0$ points of an equilateral triangle.

We are now ready to prove the main floodlight theorem.

Proof of Theorem 2.1. Suppose without loss of generality that the angles sum to exactly 2π. (Otherwise some of the angles can be decreased, and inceasing them after the plane is lit will not hurt.) Partition the angles into three sets, A_1, A_2, A_3, such that the sum of the angles in each set is at most π: take an arbitrary ordering of the angles, $\alpha_1, \ldots, \alpha_n$; let t be the maximum index such that $\sum_{i=1}^{t} \alpha_i \leq \pi$; let $A_1 = \{\alpha_1, \ldots, \alpha_t\}$; let $A_2 = \{\alpha_{t+1}\}$; and let A_3 consist of the remaining angles. Let $k_1 = t, k_2 = 1$ and $k_3 = n - t - 1$, and let $\theta_i, i = 1, 2, 3$ be the sum of the angles in A_i. By Theorem 2.3 we can find three disjoint wedges W_1, W_2, W_3 emanating from some common point such that W_i has angle θ_i and contains k_i of the given points. By Lemma 2.2 we can use the points that are in $W_i, i = 1, 2, 3$ to light the complementary wedge. Since the three complementary wedges partition the plane, this lights the whole plane.

Note that we get an $O(n \log n)$ algorithm since the initial division into A_1, A_2, A_3 is trivial, and each of the remaining two steps takes $O(n \log n)$ time.

Note also that we used an arbitrary ordering of the angles α_i to form our tripartition of the angles, thus justifying the claim that any ordering of the α_i's about a circle at infinity is realizable.

∎

3. Lighting a Stage

In this section we consider the seemingly more difficult problem of lighting a horizontal line segment, or *stage*. Given n points above the stage, and n angles, can lights of the given angles be placed at the given points to light up the stage? We do not know how to decide this question efficiently even for the special case where the n points are all on one horizontal line and the n angles are all equal.

A lamp at a given point illuminates the smallest subsegment of the stage when it projects light straight downwards, and the length of the illuminated subsegment grows monotonically as the lamp is rotated toward the horizontal. One might thus hope for the following "crossing condition": if a subsegment s can be lit by two lights from two given points, then is can be lit by "crossing" the lights, using the light at the leftmost point to illuminate the rightmost portion of s, and the light at the rightmost point to illuminate the leftmost portion of s. If this crossing condition were true then the stage light problem would be easy for equal angles, since the order of the lights hitting the stage would have to be opposite to the order of the points from which the lights emanate.

Unfortunately, the crossing condition is false. Consider the situation in Figure 3.1, where two lights of equal angles α are used. A little trigonometry will show that by having the right lamp illuminate the right portion of s and the left lamp illuminate the left portion, the length of the

illuminated segment is $(85h^2 \tan\alpha + 30h^2 \tan^2\alpha)/(12h - 40h\tan\alpha - 48h\tan^2\alpha)$, which is larger than the length of the segment illuminated by crossing the lights: $(85h^2 \tan\alpha - 30h^2 \tan^2\alpha)/(12h - 40h\tan\alpha + 7h\tan^2\alpha)$. Similar counterexamples can be provided if the two lamps are at the same height or if the left lamp is higher than the right lamp.

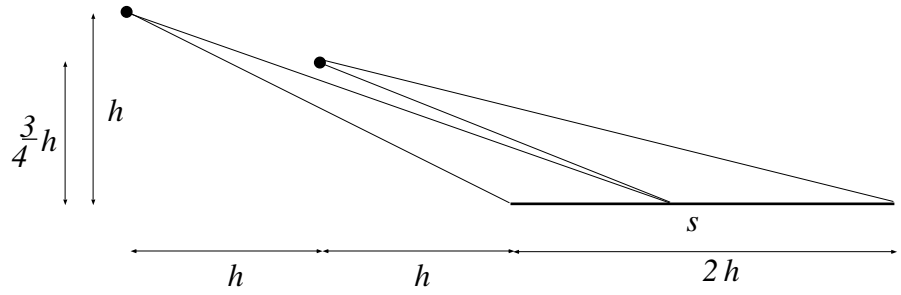

Figure 3.1

The results on lighting the plane from the previous section do have one implication for the stage light problem. Given an instance of the stage light problem, construct a line r through the right endpoint of the stage, with all the given points above r; construct a line l through the left endpoint of the stage, with all the given points above l; let θ be the angle between r and l that faces the stage. If the sum of the given angles is at least θ then by Lemma 2.2 the stage can be lit.

4. Open Questions

We have proven that lights of specified angles not exceeding π can be placed at given points to illuminate the plane if the angles sum to at least 2π. This is not true in general if one of the angles exceeds π. Is there an efficient algorithm to decide the general case: given n points and n angles, can lights of those angles be placed at the points to illuminate the plane?

Our main theorem holds because we have the freedom to assign the angles to the points and to rotate each light about its point. If the assignment of angles to lights is fixed ahead of time, then it is not always possible to rotate the lights in order to illuminate the plane even if the angles sum to 2π, and none exceeds π. Is there an efficient algorithm to decide when this is possible?

Finally, it would be of interest to have a version of the floodlight theorem for three dimensions.

Acknowledgements

Thanks to the other participants of the 1992 Workshop on Graph Theory and Computational Geometry at the Bellairs Research Institute of McGill University for fruitful discussions. A. Lubiw also thanks Jeffrey Shallit and Subhash Suri for their comments.

References

[BFP] M. Blum, R.W. Floyd, V.R. Pratt, R.L. Rivest, R.E. Tarjan. Time bounds for selection. J. Computer and System Sciences 7, 448–461, 1972.

[CRU] J. Czyzowicz, I. Rival, J. Urrutia. Galleries, light matchings, and visibility graphs. Proceedings Algorithms and Data Structures Workshop, WADS, 1989, ed. Dehne, Sack, Santoro, Lecture Notes in Computer Science 382, Springer-Verlag, 1989.

[CRCU] J. Czyzowicz, E. Rivero-Campo, J. Urrutia. Illuminating rectangles and triangles in the plane. J. Combinatorial Theory B 57, 1–17, 1993.

[CRCU2] J. Czyzowicz, E. Rivero-Campo, J. Urrutia. Optimal floodlight illumination of stages, Proceedings 5th Canadian Conference on Computational Geometry, University of Waterloo, Waterloo, Canada, 393–398, 1993.

[E] H. Edelsbrunner, *Algorithms in Combinatorial Geometry*. Springer-Verlag, 1987.

[O'R] J. O'Rourke. *Art Gallery Theorems and Algorithms*. Oxford University Press, 1987.

[PS] F.P. Preparata and M.I. Shamos. *Computational Geometry: an Introduction*. Springer-Verlag, 1985.

[S] T.C. Shermer, Recent results in art galleries, Proceedings of the IEEE 80, 1384–1399, 1992.

[Tn1] T. Tokuyama and J. Nakano, Geometric algorithms for a minimum cost assignment problem, Proceedings 7th Annual Symposium on Computational Geometry, 262–271, 1991.

[Tn1] T. Tokuyama and J. Nakano, Efficient algorithms for the Hitchcock transportation problem, Proceedings 3rd ACM-SIAM Symposium on Discrete Algorithms, 175–184, 1992.