Empty Monochromatic Triangles[*]

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Abstract

We consider a variation of a problem stated by Erdös and Guy in 1973 about the number of convex k-gons determined by any set S of n points in the plane. In our setting the points of S are colored and we say that a spanned polygon is monochromatic if all its points are colored with the same color.

As a main result we show that any bi-colored set of n points in $\mathbb{R}^2$ in general position determines a super-linear number of empty monochromatic triangles, namely $\Omega(n^{5/4})$.

1 Introduction

Erdös and Guy [9] asked the following question. "What is the least number of convex k-gons determined by any set of n points[1] in the plane?" The trivial solution for the case $k = 3$ is $\binom{n}{3}$. In addition, if we require the triangles to be empty then Katchalski and Meir [12] showed that for all $n \geq 3$ a lower bound is given by $\binom{n-1}{2}$ and that there exists a constant $c > 0$ such that there exist sets with at most cn^2 empty triangles. Around the same time Bárány and Füredi [3] showed that any set of n points has at least $n^2 - O(n \log n)$ empty triangles and they also gave examples with at most $2n^2$ empty triangles if n is a power of 2.

Valtr [17] described a configuration of n points related to Horton sets [11] with fewer than $1.8n^2$ empty triangles and also provided examples for the number of empty k-gons, e.g. with at most $2.42n^2$ empty quadrilaterals. Later Dumitrescu [8] improved the construction for triangles to $\approx 1.68n^2$, which then consequently was further improved by Bárány and Valtr [4] to $\approx 1.62n^2$, the currently best bound. It is still unknown whether the constant could be smaller than 1, that is, whether there exists a family of n-element sets with fewer than n^2 empty triangles.

We consider a related problem, where the points of the given set S are colored. A polygon spanned by points in S is called *monochromatic* if all its points are colored with the same color. In contrast to the above described race for the best constant in the uncolored case, for the number of empty monochromatic triangles for bi-colored point sets even the asymptotic

[1]Throughout, all considered point sets are in general position, that is, they do not contain three collinear points.

behavior is still not determined. See e.g. Chapter 8 of [5] for a number of similar problems on colored point sets.

A result in this direction was obtained by Devillers et al. [6]. They proved that any bi-colored point set in the plane exhibits at least $\lceil\frac{n}{4}\rceil - 2$ interior disjoint empty monochromatic triangles. In a generalization Urrutia [16] showed that in any 4-colored point set in $\mathbb{R}^3$ there is at least a linear number of interior disjoint monochromatic tetrahedra.

One might be also interested in the minimum number of colors so that we can color any given set S of n points in a way such that S does not determine an empty monochromatic triangle (or in general an empty monochromatic convex k-gon). In [6] (Theorem 3.3) this question has been settled by showing that already for three colors there are sets not spanning any empty monochromatic triangle.

The remaining question is to determine the asymptotic behavior of the number of empty monochromatic triangles for bi-colored sets. We show that any bi-colored set of n points in $\mathbb{R}^2$ in general position determines $\Omega(n^{5/4})$ empty monochromatic triangles. To the best of our knowledge no non-trivial bounds have been known before.

2 Lower Bound Construction

We start with a technical lemma which shows that for point sets with a triangular convex hull there exists a triangulation such that a sufficient fraction of its triangles are incident to vertices of the convex hull.

Lemma 1. *Let S be a set of n points in general position in the plane with 3 extreme points, that is, with a triangular convex hull and $m = n - 3$ interior points. Then S can be triangulated such that at least $m + \sqrt{m} + 1$ triangles have (at least) as one of their vertices an extreme point of S.*

Proof. Let Δ be the convex hull of S, $E(\Delta)$ the edges of Δ, and $M = S \setminus \Delta = \{q_1, ..., q_m\}$ the interior points of S, $|M| = m = n - 3$.

We first define a partial order $\leq_e$ on the elements of M. Two points $p_1, p_2 \in M$ are comparable with respect to an edge $e \in E(\Delta)$ if the open triangle formed by e and p_1 is contained in the closed triangle formed by e and p_2 ($p_1 \leq_e p_2$) or vice versa ($p_2 \leq_e p_1$), see Figure 1 for an example.
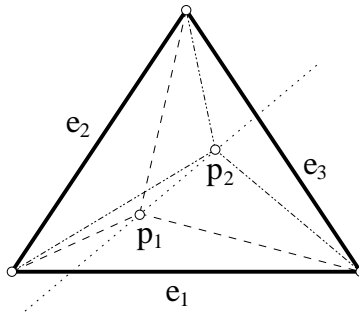

Figure 1: Two points that are comparable w.r.t. e_1 ($p_1 \leq_{e_1} p_2$) and e_3 ($p_2 \leq_{e_3} p_1$), but incomparable w.r.t. e_2.

Observe that two fixed points $p_1, p_2 \in M$ are comparable w.r.t. exactly 2 out of 3 edges of Δ. This can be seen by considering the supporting line of the edge p_1, p_2, see Figure 1. Two points are comparable w.r.t. an edge e of Δ if and only if this supporting line intersects e. This implies that if two points are not comparable w.r.t. e then they are comparable w.r.t. both other edges.

A chain is an ordered set of (pairwise) comparable points of M and an anti-chain is a set of pairwise incomparable points of M. By Dilworth's Theorem [7] there exists a chain or an anti-chain in M w.r.t. a given edge e of Δ of size $\sqrt{m}$. Because an anti-chain for e is a chain for the other two edges of Δ, we may assume w.l.o.g. that there exists a chain $q_{i_1} \leq_e \cdots \leq_e q_{i_{\sqrt{m}}}$ w.r.t. e.
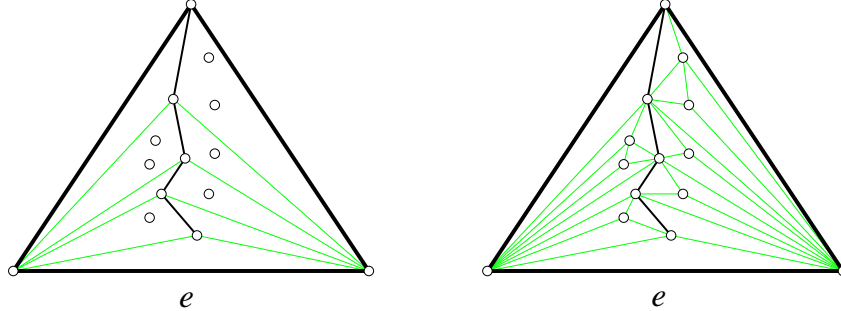

Figure 2: Left: A chain w.r.t. e (oriented upwards) and its triangulation. Right: Extended triangulation.

We obtain a triangulation of $\Delta \cup \{q_{i_1}, \ldots, q_{i_{\sqrt{m}}}\}$ by joining each q_{i_j}, $1 \leq j < \sqrt{m}$, to $q_{i_{j+1}}$ and to the end-points of e, and $q_{i_{\sqrt{m}}}$ to the vertices of Δ, see Figure 2 left. There are $2\sqrt{m} + 1$ triangles in this triangulation and all of them have at least one vertex on the convex hull. We now extend the triangulation to cover the remaining points. For each point q_i not in the chain there is precisely one end-point p of e visible to q_i and we add the edge joining q_i and p.

We have, so far, a collection of pairwise non-crossing edges, and we complete this to a triangulation of $\Delta \cup \{q_1, \ldots, q_m\}$, see Figure 2 right. There are $2\sqrt{m} + m - \sqrt{m} + 1 = m + \sqrt{m} + 1$ triangles in this triangulation with at least one of its vertices on the convex hull. $\qquad \square$

We now generalize the above result to sets with larger convex hulls. Let $CH(S)$ denote the set of vertices of the convex hull of S and $|CH(S)|$ its cardinality, that is, the number of extreme points of S.

Lemma 2. *(Order Lemma) Let S be a set of n points in general position in the plane with $h = |CH(S)|$ extreme points. Then S can be triangulated such that at least $n + \sqrt{n-h} - 2$ triangles have (at least) as one of their vertices an extreme point of S.*

Proof. Consider an arbitrary triangulation of the h convex hull points of S (ignoring inner points). Let $\tau_1, \ldots, \tau_{h-2}$ be the obtained triangles and let s_i be the number of points of S interior to τ_i. By Lemma 1 each triangle τ_i can be triangulated such that at least $s_i + \sqrt{s_i} + 1$ triangles have one of its vertices on the convex hull of τ_i and therefore on the convex hull of S. Taking the sum over all τ_i we have: $\sum_{i=1}^{h-2}(s_i + \sqrt{s_i} + 1) = \sum_{i=1}^{h-2} s_i + \sum_{i=1}^{h-2} \sqrt{s_i} + \sum_{i=1}^{h-2} 1 = (n-h) + \sum_{i=1}^{h-2} \sqrt{s_i} + (h-2) \geq n + \sqrt{\sum_{i=1}^{h-2} s_i} - 2 = n + \sqrt{n-h} - 2$. $\qquad \square$

For the next result we consider bi-colored sets. We will show that if the cardinality of the two color classes differs significantly then this implies the existence of a large number of empty monochromatic triangles.

Lemma 3. *(Discrepancy Lemma) Let S be a set of n points in general position in the plane, partitioned in a red set R and a blue set B with $|R| = |B| + \alpha$, $\alpha \geq 2$. Then S determines at least $\frac{(\alpha-2)}{6}(n+\alpha)$ empty monochromatic triangles.*

Proof. Consider a red point $r \in R$ and the star connecting r to all vertices $R \setminus r$. Completing this star to a triangulation of R gives at least $|R| - 2$ triangles having r as a vertex. At least $\alpha - 2$ of these triangles are empty of points from B, as $|B| = |R| - \alpha$. Repeating this process for all points in R we obtain at least $\frac{(\alpha-2)}{3}|R| = \frac{(\alpha-2)}{3}\frac{n+\alpha}{2} = \frac{(\alpha-2)}{6}(n+\alpha)$ empty red triangles, since we over-count a triangle at most 3 times. $\square$

Note that for the monochromatic case the Discrepancy Lemma implies the $\Omega(n^2)$ bound on the number of empty triangles given in [12], although the constants are slightly worse.

We are now ready to prove our main result.

Theorem 1. *Any bi-colored set of n points in the plane in general position determines $\Omega(n^{5/4})$ empty monochromatic triangles.*

Proof. The general idea behind the proof is to iteratively peel a monochromatic convex layer of the point set. For each layer we use the Order Lemma to obtain roughly $\sqrt{n}$ empty monochromatic triangles. If at any moment the difference of the cardinality of the two color classes is too large we utilize the Discrepancy Lemma and terminate the process. Otherwise we stop after at most $\frac{1}{3}n^{3/4}$ steps.

Let S_1 be the given bi-colored set of n points, with R_1 the set of red and B_1 the set of blue points. Let $\tilde{n} = \frac{n}{6}$. For each iteration step we construct smaller sets $S_{i+1} \subset S_i$, $R_{i+1} \subseteq R_i$, and $B_{i+1} \subseteq B_i$, respectively, with $S_{i+1} = R_{i+1} \cup B_{i+1}$. As an invariant we will have that in any step $|S_i| \geq 2\tilde{n}$ holds. The iteration stops either if at some step the discrepancy between the two sets is larger than $\tilde{n}^{1/4}$ or after at most $\frac{1}{3}n^{3/4}$ steps.

Consider the i-th step of the iteration and w.l.o.g. let $|R_i| \geq |B_i|$. There are two possible cases.

(a) If $|R_i| - |B_i| \geq \tilde{n}^{1/4}$ we apply the Discrepancy Lemma and get at least $\frac{(\tilde{n}^{1/4}-2)}{6}(2\tilde{n}+\tilde{n}^{1/4}) = \Omega(n^{5/4})$ empty monochromatic triangles.

(b) Otherwise build the convex hull of the red points and let $B_i' \subseteq B_i$ be the blue points outside of this convex hull. We denote by $r_i = |R_i|$ and $b_i = |B_i \setminus B_i'|$. We have $r_i \geq \tilde{n}$ by our invariant assumption, $r_i \geq b_i$, and $r_i \leq b_i + \tilde{n}^{1/4}$, as otherwise we apply the Discrepancy Lemma to $R_i \cup B_i \setminus B_i'$ and terminate the iteration with $\Omega(n^{5/4})$ empty monochromatic triangles as above. Note that the latter inequality implies that $|B_i'| < \tilde{n}^{1/4}$.

We apply the Order Lemma to R_i and get at least $r_i + \sqrt{r_i - |CH(R_i)|} - 2$ monochromatic (red) triangles which are by construction a subset of a triangulation of R_i incident to $CH(R_i)$. At most b_i of these triangles may contain a blue point, so we get at least $r_i - b_i + \sqrt{r_i - |CH(R_i)|} - 2 \geq \sqrt{r_i - |CH(R_i)|} - 2$ empty monochromatic triangles.

Now we show that $|CH(R_i)| < 2\tilde{n}^{1/4}$. Assume to the contrary that $|CH(R_i)| \geq 2\tilde{n}^{1/4}$ and consider the set $(R_i \setminus CH(R_i)) \cup (B_i \setminus B_i')$. This set has at most $r_i - 2\tilde{n}^{1/4}$ red points and $b_i \geq r_i - \tilde{n}^{1/4}$ blue points, so the difference is at least $\tilde{n}^{1/4}$ and as above we apply the Discrepancy Lemma and terminate.

Thus, if we don't terminate, in step i we get at least $\sqrt{\tilde{n} - 2\tilde{n}^{1/4}} - 2 \geq \frac{\sqrt{\tilde{n}}}{2}$ empty monochromatic triangles with at least one vertex in $CH(R_i)$. Note that the last inequality holds for sufficiently large $\tilde{n}$.

For the next iteration step let $R_{i+1} = R_i \setminus CH(R_i)$, $B_{i+1} = B_i \setminus B_i'$, and $S_{i+1} = R_{i+1} \cup B_{i+1}$. Note that all the empty monochromatic triangles we have constructed in step i had at least one vertex in $CH(R_i)$, that is, we will not use these vertices for the next iterations, and therefore we do not over-count.

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The process ends either by applying the Discrepancy Lemma, or after $\frac{1}{3}n^{3/4}$ steps. As in each step we obtain at least $\frac{\sqrt{\tilde{n}}}{2}$ empty monochromatic triangles, we get in both cases a total of $\Omega(n^{5/4})$ empty monochromatic triangles.

It remains to show that the invariant $|S_i| \geq 2\tilde{n}$ holds. In step i we remove $|B_i'| + |CH(R_i)| < \tilde{n}^{1/4} + 2\tilde{n}^{1/4} = 3\tilde{n}^{1/4}$ points. Thus after $\frac{1}{3}n^{3/4}$ steps we have at least $n - \frac{1}{3}n^{3/4} \cdot 3\tilde{n}^{1/4} \geq 2\tilde{n}$ points left. $\qquad\square$

3 Conclusions and Open Problems

We have not been able to construct a point set with $o(n^2)$ empty monochromatic triangles. Usually Horton sets are a good candidate to provide minimal examples with respect to determining empty convex polygons. But it turns out that every two-coloring of a Horton set has $\Omega(n^2)$ empty monochromatic triangles. A brief sketch of that fact looks as follows. Take any bi-coloring of the Horton set and note that the upper and lower part must have a linear number of red and blue points, as otherwise by the Discrepancy Lemma there would be a quadratic number of empty monochromatic triangles. Now take any triangle of three consecutive points in the upper part which form a cap. Any edge of this triangle, where at least one is monochromatic, say red, and any point from the lower part spans an empty triangle. Thus, together with the $\Theta(n)$ red points from below, it forms a linear number of empty red triangles. Since there is a linear number of such caps, we get a quadratic number of empty monochromatic triangles for the Horton set.

Other interesting sets with $O(n^2)$ empty triangles, which are not based on Horton sets, can be found in the constructions of Katchalski and Meir [12].

Considering results in [4] and [15] one can see that in the uncolored case and for sufficiently large n there is always a quadratic number of empty triangles and empty convex quadrilaterals. Moreover, there is at least a linear number of empty pentagons, but the correct bound seems to be quadratic. The status for empty convex hexagons was a long-standing open problem: Does there always exist an empty convex hexagon for sufficiently large point sets? This question has been answered only recently in the affirmative. Nicolás [13] and Gerken [10] showed independently that for large enough sets there always exists at least one empty convex hexagon. This of course implies that there exists a linear number of them, but the best upper bound is again quadratic. Finally, no empty convex 7-gons may exist. So it seems that either none, or a quadratic number of empty k-gons exists, and we believe that this might translate to colored point sets. We therefore state the following conjecture.

Conjecture 1. *Any bi-colored set of n points in $\mathbb{R}^2$ in general position determines a quadratic number of empty monochromatic triangles.*

In fact we did not obtain a single family of sets where the asymptotics of the number of empty triangles and empty monochromatic triangles differ. What we have been able to construct are sets which have 5 times fewer empty monochromatic triangles than empty triangles. The idea behind the construction is to start with a set S of n points with $t(S)$ empty triangles. W.l.o.g. S has no two points on a horizontal line. We then add a copy of S which is shifted horizontally to the right by some sufficiently small ε and color the points of S red and their duplicates blue. For each ε-near pair we get $2n - 2$ empty bi-chromatic triangles. For each empty triangle in S we get 3 new bi-chromatic triangles (not using an ε-near pair of points), but only one empty monochromatic triangle. Thus the ratio of empty triangles to monochromatic ones is $4 + \frac{2n^2 - 2n}{t(S)}$. Taking the sets constructed by Bárány and Valtr [4] with $t(S) \approx 1.62n^2$ empty triangles gives a factor of ≈ 5.23.

Another interesting question is to consider empty monochromatic convex k-gons for $k > 3$. Devillers et al. [6] (Theorem 3.4) showed that for $k \geq 5$ and any n there are bi-colored sets where no empty monochromatic convex k-gon exists. Recently in [1] it has been shown that any sufficiently large bi-colored point set contains at least one empty monochromatic, not necessarily convex, quadrilateral.

For the remaining case of empty monochromatic convex quadrilaterals it is known that they always exist in any bi-colored Horton set for $n \geq 64$ [6]. This leads to Conjecture 3.1 in [6] which states that for sufficiently large n any bi-colored set contains at least one monochromatic convex quadrilateral.

Recently, during the preparation of the journal version of this paper, and based on our work, the lower bound on the number of empty monochromatic triangles has been improved to $\Omega(n^{4/3})$ [14].

Let us finally mention that in a forthcoming paper we have been able to prove an analogous lower bound on the number of empty monochromatic simplices in $\mathbb{R}^d$, namely that any bi-colored set of n points in $\mathbb{R}^d$ determines $\Omega(n^{d-3/4})$ empty monochromatic simplices [2].

4   Acknowledgment

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