Straight Line Embeddings of Planar Graphs on Point Sets

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Abstract

Given a finite point set P_n of n points in the plane in general position, we say that P_n supports an n vertex planar graph G if there is a rectilinear embedding of G such that all the vertices of G lie on the elements of P_n. G is called universal if any point set P_n supports it. In this paper we prove that the set of universal graphs is exactly the set of outerplanar graphs. We also give an $O(n^2)$ time algorithm that produces planar embeddings of outerplanar graphs on point sets.

Keywords: Universal graphs; finite points in the plane; geometric graphs

1 Introduction

Let P_n be a set of points on the plane in general position, and G a graph with n vertices. We say that P_n supports G if there is an embedding of G on the plane in such a way that the vertices of G are mapped to the elements of P_n, and its edges to non-intersecting open straight line segments joining pairs of elements of P_n which correspond to pairs of adjacent vertices in G. We call any such embedding a *straight-line embedding* of G on P_n.

In 1990, Perles introduced the problem of embedding rooted trees on point sets with the root of the tree located at a specific element of the point set. In [3] it is shown that this embedding is always possible. Further results on this topic are presented in [4] and [1] where an optimal $O(n \ln n)$ time algorithm to obtain such embeddings is presented. A graph G is called *outerplanar* if there is a straight line embedding of G on the vertices of a convex polygon.

In this paper, we extend the results in [3] as follows: We call a planar graph G on n vertices a *universal graph* if *any* n point set supports G. By the results in [3] it follows that trees are universal. Notice that if we choose P_n to be the set of vertices of a convex polygon, it follows right away that if a graph G is universal, then it is outerplanar. In this paper we prove that the set of universal graphs is exactly the set of outerplanar graphs. We also give an algorithm to produce embeddings of n vertex planar graphs on n point set P_n in general position in $O(n^2)$ time.

2 Terminology and definitions

An embedding of a graph G on the plane is called a *straight-line* embedding if all the edges of G are represented by line segments. A graph G is called outerplanar if there is a straight-line embedding of G on the vertices of a convex polygon. All point sets considered here will be assumed to be in general position and P_n will be used to denote such sets.

3 Outerplanar graphs are universal

In this section we prove the following result:

Theorem 1 *The set of universal graphs is exactly the set of outerplanar graphs.*

Some lemmas and results will be needed to prove Theorem 1.

Without loss of generality, we will assume that G is a maximal outerplanar graph. We further assume that the vertices of G are labeled $v_0, \ldots, v_{n-1}$ such that v_i is adjacent to v_{i+1}, $i = 0, \ldots, n-1$ addition taken mod n, i.e. the unique Hamiltonean cycle of G is given by the ordered labeling of its vertices. Edge $v_i - v_{i+1}$ will be called an external edge of G, $i = 0, \ldots, n-1$. Consider a triangle (u, v, w) induced by three mutually adjacent vertices u, v, w of G. We say that (u, v, w) is an external triangle of G if at least one of its edges is on the Hamiltonean cycle of G; we call (u, v, w) an $(r, s)-$*triangle* if the components of $G - \{u, v, w\}$ have r and s vertices respectively. We will allow either one of r or s to be 0; this allows us to cover the case when (u, v, w) contains two consecutive edges of the Hamiltonean cycle of G.

Consider a n point set P_n, two integers r and s such that $r + s = n - 3$, and a triangle $t(a, b, c)$ with a and b consecutive vertices of $Conv(P_n)$, and $c \in P_n$. We say that $t(a, b, c)$ is an $(r, s) - triangle$ of P_n if:

I. No element of P_n lies in the interior of $t(a, b, c)$

II. There is a line l through c that intersects the interior of $t(a, b, c)$ such that there are r elements of $P_n - \{a, b, c\}$ on the same side of l as a, and s elements of $P_n - \{a, b, c\}$ on the other side of l. See Figure 1.

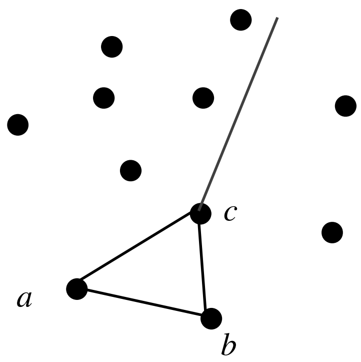

Figure 1: A (6,2) triangle of P_8

We denote by $l(a)$ the subset of elements of P_n on the same side of l as a and including also point c. Similarly, we define $l(b)$.

Lemma 1 *Let a and b be two consecutive points in $Conv(P_n)$, and two integers r and s such that $r + s = n - 3$. Then there is a point c in P_n such that $t(a, b, c)$ is an $(r, s) - triangle$ of P_n.*

Proof: Consider all the points of P_n such that each of them together with a and b are the vertices of a triangle containing no point of $P - n$ in its interior. Assume that these points are labeled $u_1, \ldots, u_k$ in the counter-clockwise direction around a. Associate to each u_i a weight w_i equal to the number of points in $P_n - \{a, b, u_i\}$, to the right of the line connecting a to u_i, $i = 1, \ldots, k$. See Figure 2.

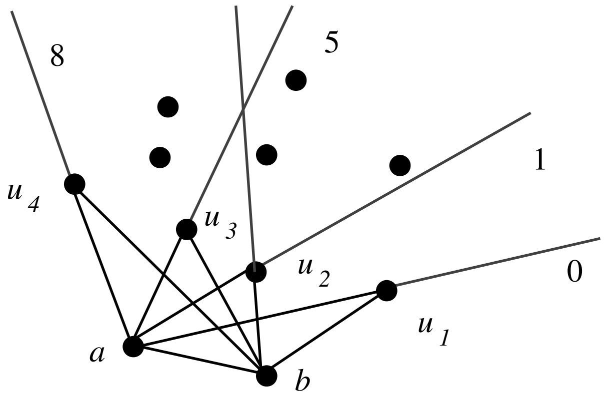

Figure 2:

Let m be the index such that $w_{m-1} < s$ and $s \leq\ w_m$. If $w_m = s$ take $c = u_m$ and let l be a line through u_m obtained by a slight counter-clockwise rotation around u_m of the line joining a to u_m.

Suppose then that $s < w_m$. We observe now that the number of points to the right of the line joining b to u_{m-1} is at least $w_m - 1$. Furthermore, the triangle bounded by the line segment $\overline{ab}$ and the lines joining a and b to u_m contains no element of P_n in its interior, otherwise u_{m-1} and u_m would not be consecutive vertices that generate empty triangles with a and b! It now follows that there is a line through u_{m-1} intersecting the interior of the triangle $t(a, b, u_{m-1})$ leaving exactly s elements of $P_n - \{a, b, u_{m-1}\}$ to its right. Our result now follows. In Figure 2 we show the case when $s = 4$ and $m = 3$.

Proof of Theorem 1. Let G be a universal graph and P_n be the set of vertices of a convex polygon. Since G is universal, P_n supports G; hence there is a planar straight-line embedding of G on P_n and G is outerplanar. We now prove that any point set P_n supports any outerplanar graph G. Recall that the vertices of G are labeled $\{v_0, \ldots, v_{n-1}\}$ such that v_i is adjacent to v_{i+1}, $i = 0, \ldots, n-1$ addition taken mod n. We will actually prove an even stronger result, i.e. we prove that given two consecutive points a and b in $Conv(P_n)$ and any external edge $v_i - v_{i+1}$ of G, there is an embedding of G on P_n such that v_i lies on a and v_{i+1} lies on b.

Our result holds trivially if G has three vertices. Consider now any outerplanar graph on n vertices and any n point set P_n. Take an external edge $v_i - v_{i+1}$ of G and two consecutive vertices a and b in $Conv(P_n)$. Since G is outerplanar, there is a unique vertex v_k of G adjacent to both v_i and v_{i+1}. Thus (v_i, v_{i+1}, v_k) is an external triangle of G. Assume without loss of generality that $i+1 < k$ and that $k-(i+1)-1 = s$. Then the components of $G-\{v_i, v_{i+1}, v_k\}$ have cardinalities r and s respectively such that $r+s = n-3$ and (v_i, v_{i+1}, v_k) is an $(r,s) - triangle$ of G. See Figure 3.

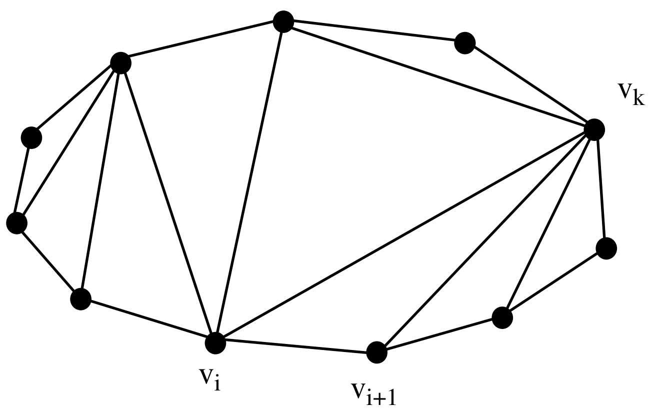

Figure 3:

By Lemma 1, there is an $(r,s) - triangle\ t(a,b,c)$ of P_n. Let l be a line through c as in II and consider the subsets $l(a)$ and $l(b)$ of P_n as defined above. Notice that a and c lie on $Conv(l(a))$ and b and c lie on $Conv(l(b))$. Let H_1 and H_2 be the subgraphs of G induced by $\{v_k, \ldots, v_i\}$ and $\{v_{i+1}, \ldots, v_k\}$ respectively, addition taken mod n. Clearly, $v_i - v_k$ and $v_k - v_{i+1}$ are external edges of H_1 and H_2 respectively. Thus by induction there is an embedding of H_1 on $l(a)$ such that v_i lies on a and v_k lies on c. Similarly there is an embedding of H_2 on $l(b)$ such that v_{i+1} lies on b and v_k lies on c. Combining these embeddings of H_1 and H_2 we can obtain an embedding of G on P_n such that the edge $v_i - v_{i+1}$ lies on the line segment joining a and b. Our result now follows.

We now present an algorithm to obtain rectilinear embeddings of outerplanar graphs on point sets in $O(n^2)$ time.

In our initial step, for each element a of P_n we sort the slopes of all the edges connecting a to all other elements of P_n. This can be done in $O(n^2)$ using well known techniques in computational geometry [2]. Assume then that these orders are available at all the elements of P_n. We now show that having these orders available, we can implement the construction of Lemma 1 in linear time. This will prove our result.

As in the proof of Lemma 1, let $\{u_1, \ldots, u_k\}$ be the elements of $P_n - \{a, b\}$ that form empty triangles with a and b. We now prove:

Lemma 2 *Given two consecutive points a and b in $Conv(P_n)$, and two integers r and s such that $r + s = n - 3$ we can find an $(r, s) - triangle$ $t(a, b, c)$ of P_n in linear time.*

Proof: We show first that $\{u_1, \ldots, u_k\}$ and $\{w_1, \ldots, w_n\}$ as described in Lemma 1 can be found in linear time. Let p be any element of $P_n - \{a, b\}$. Assign to p two integer coordinates, corresponding to the position of p when sorting the elements of $P_n - \{a, b\}$ around a and b in the counter-clockwise and clockwise direction respectively. For example point u_2 in Figure 2 would receive coordinates $(2, 5)$. This maps the points of $P_n - \{a, b\}$ to points on the plane with integer coordinates, and points generating empty triangles correspond to minimal elements under vector dominance. Using techniques in [5] it follows that $\{u_1, \ldots, u_k\}$ can be found in linear time. We now notice that if u_i is in position k in the sorted order of the elements of $P_n - \{a, b\}$ around a, w_i is exactly $k - 1$. Using this, we can now easily determine c, l, $l(a)$ and $l(b)$ as in Lemma 1 in linear time. Our result follows.

We now have:

Theorem 2 *Given a maximal outerplanar graph G and a point set P_n we can find an embedding of G on P_n in $O(n^2)$ time.*

We believe that our algorithm is not optimal, and that an $O(n \log n)$ time algorithm exists.

References

[1] P. Bose, M. McAllister, and J. Snoeyink, Optimal algorithms to embed trees in a point set, *In Graph Drawing* 1995.

[2] H. Edelsbrunner and L. Guibas, Topologically sweeping an arrangement. *Proc. 4th ACM Symposium. Computational Geometry* 1988, 44-55.

[3] Y. Ibeke, M. Perles, A. Tamura, and S. Tokunaga, The rooted tree embedding problem into points on the plane. *Discrete and Computational Geometry*, 11:51-63, 1994.

[4] J. Pach and J. Torocsik, Layout of rooted trees. Technical report CS-TR-369-92, Princeton, Feb. 1992.

[5] F. P. Preparata and M. I. Shamos. *Computational Geometry: An introduction*. Springer-Verlag, New York 1985.