Weak separators, vector dominance, and the dual space

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Abstract. In this paper we study two problems related to vector dominance and rectilinear separators of point sets on the plane. We show that the best *weak separator* of a set of bicolored points on the plane can be obtained in $O(n^2)$ time. We also study some problems arising from the rectilinear convex hull of point sets, but in the dual space. This produces some attractive geometric visualizations of staircases and rectilinear separators in that space.

Introduction

In this paper we study two problems arising from the study of the rectilinear convex hull of point sets on the plane. Without loss of generality, and to make our presentation easier, we will assume that all the points in our point sets have positive coordinates.

A *quadrant* of the plane is the intersection of two half-planes whose supporting lines are parallel to the x- and y-axes. Let S be a set of points in the plane in general position. We say that a quadrant is S-*free* if its interior contains no point in S.

The *rectilinear convex hull* of a point set S is defined as

$$\mathcal{RH}(S) = \mathbb{R}^2 - \bigcup_{Q \text{ is an } S\text{-free quadrant}} Q.$$

Observe that, if we rotate the plane around the origin, the rectilinear convex hull of a point set changes. The problem of finding a rotation of the plane that produces a rectilinear convex hull with minimum area was studied in [**1**], where an $O(n^2)$ time algorithm to solve this problem was presented. Their algorithm was improved to $\Theta(n \log n)$ in [**2**].

The rectilinear convex hull of a point set was first studied in [**3**]. A point $p = (a, b) \in S$ is dominated by $q = (c, d) \in S$, $p \neq q$, if $a \leq c$ and $b \leq d$. A polygonal curve C is called *rectilinear* if it consists of a sequence of line segments each of which is horizontal or vertical, and C is called a *staircase* if it is monotone with respect to the x- and y-axes. In the rest of this paper, we will further assume that a staircase is monotonically decreasing with respect to the x-axis.

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Let $S = R \cup B$ be a set of n points on the plane in general position such that the elements of R and B are colored red and blue respectively. In this paper, we are interested in the following problem: Find a staircase that best *classifies* R and B; that is, find a staircase C such that the number of red points below it plus the number of blue points above it is maximized. Such a staircase we called the best *weak staircase separator*. We obtain an $O(n^2)$-time algorithm to solve this problem.

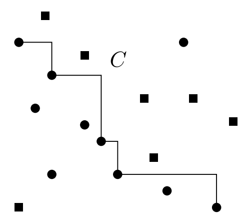


FIGURE 1. A staircase polygonal weak separator C separating red points (●) and blue points (■).

While solving the above problem, we stumbled on the following problem: Can we give an interpretation of the rectilinear convex hull of a set of points in the dual space? What about the concept of rectilinear separability?

Recall that the dual of a point $p = (a, b)$ of the plane, denoted by ℓ_p, is the non-vertical line with equation $y = ax - b$. The dual of ℓ_p is p. It is well known that, under duality, collinear points are mapped to sets of concurrent lines, and concurrent lines are mapped to collinear points [**4, 5**].

In Section 2, we study this problem, and show an attractive interpretation of the rectilinear convex hull of a point set. We will assume that our point sets are contained in the positive quadrant of the plane, and show that the rectilinear convex hull of a point set looks like a set of rays emanating from a *sun*.

1 Computing the best staircase weak separator

In this section, we outline our algorithm to obtain a best staircase weak separator. Our algorithm is based on dynamic programming. We perform first an $O(n \log n)$ preprocessing on $R \cup B$, and then perform a line sweep from left to right, stopping at every point of R. It is easy to see that the best staircase weak separator can be chosen in such a way that it is determined by a set of points in R, which are the right endpoints of the horizontal lines of the staircase. For every point r_i in R, we maintain the optimal weak separator whose rightmost vertex is precisely r_i.

Let us assume that the elements of S are sorted from left to right according to their x-coordinate, and that the elements of R are labelled $\{r_1, \ldots, r_m\}$ in such a way that, if $i < j$, the point r_i is to the left of r_j. This labeling can be achieved in $O(n \log n)$ time. Recall that using quadratic preprocessing on S [**6**], we can find in constant time the number of red and blue points of S within any isothethic rectangle. We now sweep a vertical line from left to right stopping at all the points in R.

For each point r_j in R, we find in $O(n)$ time the point r_i such that the optimal weak separator whose last two vertices are r_i and r_j. We can do this in $O(n)$ time since, for each r_k, $k < j$, we can calculate in constant time the number of red and blue points dominated by r_j that are not dominated by r_k. Due to lack of space, the proof of the correctness of our algorithm is omitted. Our algorithm works in $O(n^2)$ time. Thus we have the following result:

Theorem 1.1. *An optimal staircase weak separator of S can be calculated in $O(n^2)$ time.*

Suppose now that we can rotate the plane. We would like to find an angle θ such that, when we rotate the plane θ degrees around the origin, we obtain the best weak separator over all $\theta \in [0, 2\pi)$. An immediate corollary of Theorem 1.1 is that we can find the best unoriented weak separator in $O(n^4)$ time by getting the best oriented weak separators in each of the $\binom{n}{2}$ combinatorially distinct directions of S, and choosing the best one of all. We believe that finding the angle θ that produces the best unoriented weak separator can be done in $O(n^3 \log n)$ time.

2 Vector dominance and staircases in the dual space

Consider the elements of S under the partial order defined by $(a, b) \succ (c, d)$ if and only if $a \geq c$ and $b \geq d$, $(a, b) \neq (c, d)$. Since all the lines ℓ_p in the dual space are non-vertical, the y-axis splits them into two rays. The ray to the right of y-axis will be denoted by ℓ_p^+, the one to its left ℓ_p^-, and they will be called, respectively, the *positive* and the *negative* semi-lines of ℓ_p.

Observe that a point p dominates another point s in the partial order $\succ$ if and only if the slope of ℓ_p is greater than the slope of ℓ_s, and ℓ_p intersects the y-axis below the point where ℓ_s intersects it. This implies that ℓ_p and ℓ_s intersect each other to the right of y-axis, or simply that ℓ_p^+ intersects ℓ_s^+ (Figure 2).

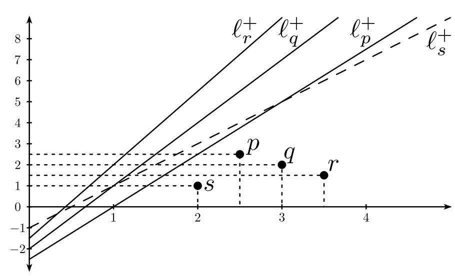
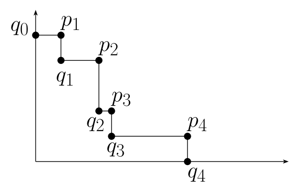


FIGURE 2. The anti-chain formed by p, q, and r, the dominated point s, and their transformations in the dual space.

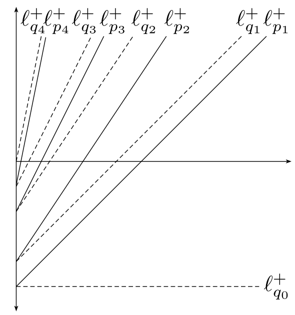
On the other hand, if two points p and q of S are not comparable in $\succ$, then ℓ_p^+ and ℓ_q^+ do not intersect. If in the dual we consider only the positive semi-lines of the dual lines of the elements of S, then we can see that an anti-chain of points in the partial order generates a set of non-intersecting rays with increasing slopes (Figure 2).

Every anti-chain $p_1, p_2, \ldots, p_k$ of S with respect to $\succ$ determines a staircase polygonal chain $\mathcal{S}$ as shown in Figure 3(a). Define points $q_0, \ldots, q_k$ on the staircase defined by $p_1, p_2, \ldots, p_k$ as in Figure 3(a). Since $p_1, \ldots, p_k$ are pairwise non comparable, $\ell_{p_1}^+, \ldots, \ell_{p_k}^+$ do not intersect each other.

If we traverse $\mathcal{S}$ from q_0 to q_k, we can see that, when we traverse the horizontal segment defined by the points q_i and p_{i+1}, in the dual space we rotate the ray $\ell_{q_i}^+$ until its slope is the same as that of $\ell_{p_{i+1}}^+$. When we traverse the vertical segment defined by



(a) An anti-chain with extra points to form a staircase.



(b) The anti-chain and extra points in the dual space.

FIGURE 3. An anti-chain $p_1, \dots, p_4$ with extra points $q_0, \dots, q_4$ and how it looks in the dual space.

the points p_i and q_i in the dual space we translate the ray $\ell^+_{p_i}$ upwards until it reaches $\ell^+_{q_i}$; see Figure 3(b).

3 Conclusions

We point out that our study of vector dominance in the dual space has allowed us to give attractive geometric interpretations of objects such as empty isothetic rectangles with opposite vertices in a fixed point set. In addition, it has enabled us to develop algorithms such as finding the rectilinear convex hull of a point set, or efficiently calculating the set of points below a staircase. These algorithms work directly in the dual space, and usually have the same complexity as those in the *primal space*. In the full version of this paper, we will explore these results in more detail.

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