Simple Euclidean arrangements with no (≥ 5)–gons

Jesús Leaños* Mario Lomelí† Criel Merino‡ Gelasio Salazar§ Jorge Urrutia¶

March 12, 2007

Abstract

It is shown that if a simple Euclidean arrangement of n pseudolines has no (≥ 5)–gons, then it has exactly $n-2$ triangles and $(n-2)(n-3)/2$ quadrilaterals. We also describe how to construct all such arrangements, and as a consequence we show that they are all stretchable.

1 Introduction

Our goal in this discussion is to analyze simple Euclidean arrangements of pseudolines in which every bounded cell is either a triangle or a quadrilateral.

We recall that a simple noncontractible closed curve in the projective plane $\mathbb{P}$ is a *pseudoline*, and an *arrangement of pseudolines* is a collection $\mathcal{B} = \{P_0, P_1, \ldots, P_n\}$ of pseudolines that intersect (necessarily cross) pairwise exactly once. Since $\mathbb{P} \setminus P_0$ is homeomorphic to the Euclidean plane $\mathbb{E}$, we may regard $\{P_1, \ldots, P_n\}$ as an *arrangement of pseudolines* in $\mathbb{E}$ (and regard $P_1, \ldots, P_n$ as *pseudolines* in $\mathbb{E}$). An arrangement is *simple* if no point belongs to more than two pseudolines.

The cell complex of an Euclidean arrangement has both bounded and unbounded cells. As in [5], we are only interested in bounded cells (whose interiors are the *faces*). Thus it is clear what is meant by a *triangle*, a *quadrilateral*, or, in general, an *n–gon* of the arrangement. In this work we are interested in arrangements in which every bounded cell is either a triangle or a quadrilateral; for obvious reasons we say that such an arrangement is (≥ 5)-*gon—free*.

One of the most interesting and widely studied problems concerning arrangements of lines and pseudolines is the determination of upper and lower bounds for the number p_k of k–sided faces. An extensive amount of research in such problems followed Grünbaum's seminal work [8] on arrangements and spreads (see for instance [6, 13, 14]). In [5], it was proved that in every simple Euclidean arrangement, $p_3 \geq n-2$. Moreover, it immediately follows from the proof of Proposition 2.1 in [5] that in every (≥ 5)-gon–free–arrangement, $p_3 = n-2$.

In this paper we describe procedure with which every (≥ 5)-gon–free simple Euclidean arrangement can be recursively constructed from the (unique up to isomorphism) simple Euclidean

*Instituto de Física, Universidad Autónoma de San Luis Potosí, San Luis Potosí, SLP, Mexico 78000. jelema@ifisica.uaslp.mx

†Facultad de Ciencias, Universidad Autónoma de San Luis Potosí, San Luis Potosí, SLP, Mexico 78000. lomeli@ifisica.uaslp.mx

‡Instituto de Matemáticas, Universidad Nacional Autónoma de México, México DF, Mexico 04510. merino@matem.unam.mx

§Instituto de Física, Universidad Autónoma de San Luis Potosí, San Luis Potosí, SLP, Mexico 78000. gsalazar@ifisica.uaslp.mx. Supported by FAI–UASLP and by CONACYT Grant 45903.

¶Instituto de Matemáticas, Universidad Nacional Autónoma de México, México DF, Mexico 04510. urrutia@matem.unam.mx. Partially supported by CONACYT, Proyecto SEP-2004-Co1-45876.

arrangement with 3 pseudolines. The central concept in this regard are twin pseudolines. Consider a simple Euclidean arrangement of pseudolines $\mathcal{B} = \{P_1, P_2, \ldots, P_n\}$, and let $P_i, P_j \in \mathcal{B}$. Then the intersection point $p_{i,j}$ between P_i and P_j divides P_i (respectively P_j) into two subarcs, say A_i and B_i (respectively A_j and B_j). We say that P_i and P_j are *twin* pseudolines (see Figure 1) if the following hold:

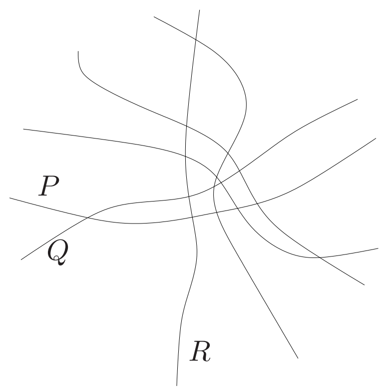

Figure 1: A simple Euclidean (≥ 5)-gon–free arrangement with 6 pseudolines. Pseudolines P and Q are twin. The triangle defined by P, Q, and R, is P–critical and Q–critical, but not R–critical.

(i) One of A_i and B_i (say B_i, without any loss of generality) does not contain any intersection point of P_i with the pseudolines in $\mathcal{B} \setminus \{P_i, P_j\}$. Similarly, one of A_j and B_j (say B_j, without any loss of generality) does not contain any intersection point of P_j with the pseudolines in $\mathcal{B} \setminus \{P_i, P_j\}$.

(ii) As we traverse each of A_i and A_j, starting at $p_{i,j}$, we intersect the pseudolines in $\mathcal{B} \setminus \{P_i, P_j\}$ in the exact same order.

Twin pseudolines are essential for the understanding of (≥ 5)-gon–free arrangements, as the following statement reveals.

Theorem 1 *In every simple (≥ 5)-gon–free Euclidean arrangement with at least 4 pseudolines there are distinct pseudolines Q_1, Q_2, Q_3, Q_4 such that Q_1 and Q_2 are twin, and Q_3 and Q_4 are twin.*

Remark 2 *Theorem 1 is best possible in every sense: (i) for each $n \geq 4$ there is a simple (≥ 5)-gon–free Euclidean arrangement $\mathcal{A}$ with n pseudolines, with exactly two pairs of twin pseudolines; and (ii) for each $n \geq 5$ there is a simple Euclidean arrangement $\mathcal{A}$ with n pseudolines, only one (≥ 5)–gon, and such that $\mathcal{A}$ has no twin pseudolines.*

The constructions that prove Remark 2 are given in Figure 2. Start by drawing P_1, P_2, P_3, P_4 as shown. Now suppose that for some $r \geq 4$, $\{P_1, P_2, \ldots, P_r\}$ has been constructed. If r is even, then add P_{r+1} so that P_{r-1} and P_{r+1} are twin, in such a way that as we traverse P_{r+1} from right to left, the first curve in $\{P_1, P_2, \ldots, P_r\}$ we intersect is P_{r-1}. If r is odd, then add P_{r+1} so that P_r and P_{r+1} are twin, in such a way that as we traverse P_{r+1} from left to right, the first curve in $\{P_1, P_2, \ldots, P_r\}$ we intersect is P_r. The simple Euclidean arrangement $\{P_1, P_2, \ldots, P_r\}$ obtained at each step is (≥ 5)-gon–free, and has exactly two pairs of twin pseudolines, namely $\{P_1, P_2\}$ and

$\{P_{r-1}, P_{r+1}\}$ if r is even, and $\{P_1, P_2\}$ and $\{P_r, P_{r+1}\}$ if r is odd. Now suppose that $\{P_1, P_2, \ldots, P_k\}$ has been constructed for some $k \geq 4$, and add a pseudoline Q (dashed pseudoline in this figure) as illustrated. Then the simple Euclidean arrangement $\{P_1, P_2, \ldots, P_k, Q\}$ has no twin pseudolines at all, and it has exactly one (≥ 5)–gon, namely the 5–gon bounded by P_1, P_2, P_{k-1}, P_k, and Q.

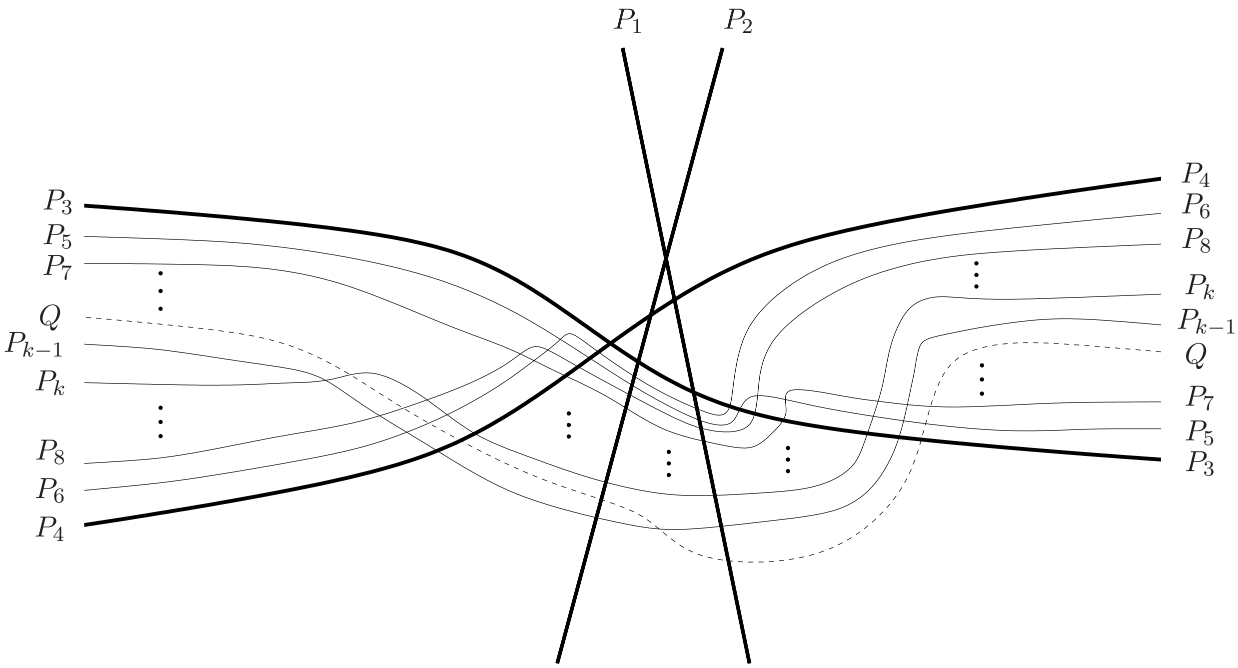

Figure 2: This construction shows that Theorem 1 is best possible (see Remark 2).

Theorem 1 not only gives a procedure to generate all simple (≥ 5)-gon–free Euclidean arrangements, but also has a neat consequence in the realm of stretchability.

An *arrangement of lines* in $\mathbb{E}$ is a collection of straight lines, no two of them parallel. Thus every arrangement of lines is an arrangement of pseudolines. On the other hand, not every arrangement of pseudolines is *stretchable*, that is, equivalent to an arrangement of lines (recall that two arrangements are *equivalent* if they generate isomorphic cell decompositions of $\mathbb{E}$). Every arrangement of 8 pseudolines is stretchable [7], but there is a simple non–stretchable arrangement of 9 pseudolines [12] (unique up to isomorphism; see [9]). Stretchability questions are typically difficult: deciding stretchability is NP–hard [16], even for simple arrangements [2].

The concept of stretchability is particularly relevant because of the close connection between arrangements of pseudolines and rank 3 oriented matroids: on this ground, the problem of stretchability of arrangements is equivalent to the problem of realizability for oriented matroids (see [1, 11]).

The following consequence of Theorem 1 settles the issue of stretchability for (≥ 5)-gon–free arrangements.

Theorem 3 *Every simple (≥ 5)-gon–free Euclidean arrangement is stretchable.*

The rest of this paper is organized as follows. Theorems 1, and 3, are proved in Sections 2, and 3, respectively. In Section 4 we show that there are exponentially many nonisomorphic (≥ 5)-gon–free arrangements.

2 Simple Euclidean (≥ 5)-gon–free arrangements have twin pseudolines: proof of Theorem 1

Before proceeding with the proof of Theorem 1, we establish a straightforward, yet essential, observation.

Claim 4 *If $\mathcal{B}$ is a simple Euclidean (≥ 5)-gon–free arrangement, then every subarrangement of $\mathcal{B}$ is also (≥ 5)-gon–free.*

Proof. If an arrangement has an r–gon D with $r \geq 5$, and we add a pseudoline P to it, then either P leaves D untouched, or divides D into two polygons, at least one of which has at least 5 sides. In either case, the augmentated arrangement also has an r–gon with $r \geq 5$. ∎

Proof of Theorem 1. First we show that there is at least one pair of twin pseudolines.

We proceed by induction on n. It is readily checked that the statement holds for the unique (up to isomorphism) simple Euclidean arrangement with 4 pseudolines. Thus we assume it holds for $n = k \geq 4$, and consider a simple Euclidean (≥ 5)-gon–free arrangement $\mathcal{B} = \{P_1, P_2, \ldots, P_{k+1}\}$.

By the inductive hypothesis and Claim 4, $\mathcal{B} \setminus \{P_{k+1}\}$ has a pair of twin pseudolines, say (without any loss of generality) P_1 and P_2. Moreover, we may also assume without any loss of generality that as we traverse P_1 (and P_2 as well, since they are twin), we meet $P_3, P_4, P_5, \ldots, P_k$ in this order. Thus, for each i, $3 \leq i \leq k-1$, P_1 and P_2 form a quadrilateral with P_i and P_{i+1}, and P_1, P_2, and P_3 form a triangle. We will refer to quadrilaterals formed by P_1, P_2, together with P_i and P_{i+1}, for some i, $3 \leq i \leq k-1$, as *basic* quadrilaterals.

Thus the layout of $P_1, P_2, \ldots, P_k$ is as illustrated in Figure 3.

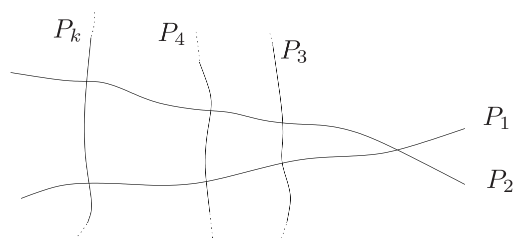

Figure 3: The pseudolines P_1 and P_2 are twin in the arrangement $\{P_1, P_2, \ldots, P_k\}$.

Suppose first that P_{k+1} crosses P_1 or P_2 to enter a basic quadrilateral. Then it must cross both P_1 and P_2 in the same quadrilateral (otherwise a pentagon would be formed, contradicting the assumption that $\mathcal{B}$ is (≥ 5)-gon–free), and so clearly P_1 and P_2 are also twin in $\mathcal{B}$. Thus for the rest of the proof we assume that P_{k+1} crosses neither P_1 nor P_2 in a basic quadrilateral.

Now suppose that P_{k+1} crosses the triangle defined by P_1, P_2, and P_3. The exchangeable role between P_1 and P_2 then allows us to assume that P_{k+1} crosses P_1 in this triangle. If P_{k+1} leaves the triangle by intersecting P_2, then we are clearly done, since then P_1 and P_2 are still twin in $\mathcal{B}$. Thus we assume that P_{k+1} leaves the triangle by intersecting with P_3. Now after intersecting P_3 (and thus entering the basic quadrilateral defined by P_1, P_2, P_3, and P_4), P_{k+1} must then intersect P_4, as otherwise a 5–gon would be formed, contradicting the assumption that $\mathcal{B}$ is (≥ 5)-gon–free. The same reasoning shows that P_{k+1} must intersect $P_4, P_5, \ldots, P_k$ in the given order. Thus P_{k+1} intersects P_2 either before intersecting P_1 or after intersecting P_k. It is straightforward to check that in either case P_2 and P_{k+1} are twin in $\mathcal{B}$. Thus for the rest of the proof we assume that P_{k+1} does not cross the triangle defined by P_1, P_2, and P_3.

The third case we analyze is when P_{k+1} intersects both P_1 and P_2 after they have already intersected P_k. That is, as we traverse P_1 we intersect the other pseudolines in the order $P_2, P_3, \ldots,$ P_k, P_{k+1}, and as we traverse P_2 we intersect the other pseudolines in the order $P_1, P_3, \ldots, P_k, P_{k+1}$. In this case P_1 and P_2 are clearly also twin in $\mathcal{B}$.

The remaining possibilities to be explored are that either (i) P_{k+1} intersects P_1 before P_2 intersects P_1; or (ii) P_{k+1} intersects P_2 before P_1 intersects P_2. Note that (i) and (ii) do not exclude each other. These conditions are equivalently described as follows: either (i) P_1 intersects $P_{k+1}, P_2, P_3, \ldots, P_k$ in the given order; or (ii) P_2 intersects $P_{k+1}, P_1, P_3, \ldots, P_k$ in the given order. Again the exchangeable roles of P_1 and P_2 allows us to assume that (i) applies. Also by exchanging P_1 and P_2 if necessary we may assume that as we traverse P_{k+1} in one of the two possible directions we intersect P_1 and the next pseudoline we intersect is P_i with $i \geq 3$. It is straightforward to check that if $i \neq 3$, then P_1, P_2, P_{k+1}, P_i, and P_3 contribute to a (≥ 5)–gon (see Figure 4). Thus $i = 3$. The same reasoning shows that the next pseudoline that P_{k+1} intersects must be P_4, and so on. Thus P_{k+1} intersects $P_1, P_3, P_4, \ldots, P_k$ in this order, and it must intersect P_2 either before intersecting P_1 or after intersecting P_k. As illustrated in Figure 5, it is readily checked that in either case P_2 and P_{k+1} are twin in $\mathcal{B}$.

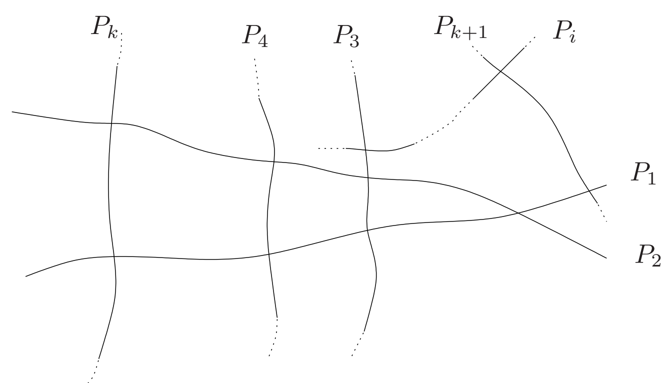

Figure 4: If P_{k+1} intersects P_i before intersecting P_3, then $P_{k+1}, P_i, P_3, P_2, P_1$ contribute to a (≥ 5)–gon.

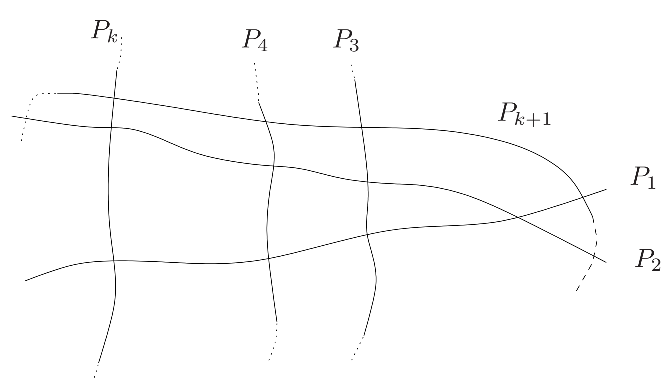

Figure 5: If P_{k+1} intersects $P_1, P_3, P_4, \ldots, P_k$ in this order, then P_{k+1} could intersect P_2 as in the dashed extension or as in the dotted extension. In either case, P_2 and P_{k+1} are twin.

We have thus proved that there is at least one pair of twin pseudolines, as claimed at the beginning of the proof.

We finally show that there are at least two distinct pairs of twin pseudolines, as claimed in Theorem 1. Again we proceed by induction on n, and the base case $n = 4$ is easily checked. Thus we assume the statement holds for $n = k \geq 4$, and consider a simple Euclidean (≥ 5)-gon–free arrangement $\mathcal{B} = \{P_1, P_2, \ldots, P_{k+1}\}$. We assume without any loss of generality that P_1 and P_2 are twin pseudolines in $\mathcal{B}$.

By the inductive hypothesis, $\mathcal{B}\setminus\{P_1\}$ has two disjoint pairs of twin pseudolines. Then at least one of these pairs, say $\{P_i, P_j\}$, is disjoint from $\{P_2\}$. Since P_i and P_j are twin pseudolines in $\mathcal{B} \setminus \{P_1\}$, then as we traverse P_2 and we intersect P_i, immediately after that (in $\mathcal{B} \setminus \{P_1\}$) we intersect P_j. On the other hand, P_1 and P_2 are twin in $\mathcal{B}$, and so P_1, P_2, P_i, and P_j must form a basic quadrilateral in $\mathcal{B}$. Therefore P_i and P_j are also twin pseudolines in $\mathcal{B}$. Since $\{P_1, P_2\} \cap \{P_i, P_j\} = \emptyset$, this completes the proof. ∎

3 Simple Euclidean (≥ 5)-gon–free arrangements are stretchable: proof of Theorem 3

We proceed inductively. The unique (up to isomorphism) simple Euclidean (≥ 5)-gon–free arrangement with 3 pseudolines is clearly stretchable. Now fix $k \geq 3$, and suppose that every simple Euclidean (≥ 5)-gon–free arrangement with k pseudolines is stretchable, and let $\mathcal{B} = \{P_1, P_2, \ldots, P_{k+1}\}$ be a simple Euclidean (≥ 5)-gon–free arrangement. By Theorem 1, there is a pair of twin pseudolines in $\mathcal{B}$, say P_k and P_{k+1} without any loss of generality.

By the inductive hypothesis, $\mathcal{B} \setminus \{P_{k+1}\}$ is stretchable, so it can be equivalently drawn with $P_1, P_2, \ldots, P_k$ as straight lines. It now suffices to observe that since P_k and P_{k+1} are twin, then P_{k+1} may be added as a straight line, if it is drawn sufficiently close to P_k. ∎

4 On the number of nonisomorphic (≥ 5)-gon–free arrangements

Our aim in this section is to show that there are exponentially many nonisomorphic (≥ 5)-gon–free arrangements.

We note that there are several types of isomorphism for pseudoline arrangements (see for instance [10]). We will explore two natural, important types of isomorphism, and show that under both criteria there are exponentially many nonisomorpohic (≥ 5)-gon–free arrangements.

4.1 Isomorphism by local sequences

In order to consider a standard, widely studied type of isomorphism, it is useful to work with the representation of arrangements illustrated in Figure 6. The type of isomorphism we now analyze arises from considering the order in which each pseudoline gets intersected by the other pseudolines (this is the *local sequence* of the pseudoline). For instance, the local sequences of the first arrangement in Figure 6 are $\ell(P_1) = P_2P_3$, $\ell(P_2) = P_1P_3$, and $\ell(P_3) = P_1P_2$, whereas the local sequences for the second arrangement are $\ell(P_1) = P_2P_3$, $\ell(P_2) = P_1P_3$, and $\ell(P_3) = P_1P_2$. Two arrangements are *isomorphic by local sequences* (for brevity, just *isomorphic* throughout this subsection) if and only if they have the same local sequences. If we regard arrangements as reflection networks, this isomorphism corresponds to the equivalence relation defined by Knuth [9] (see also [4]).

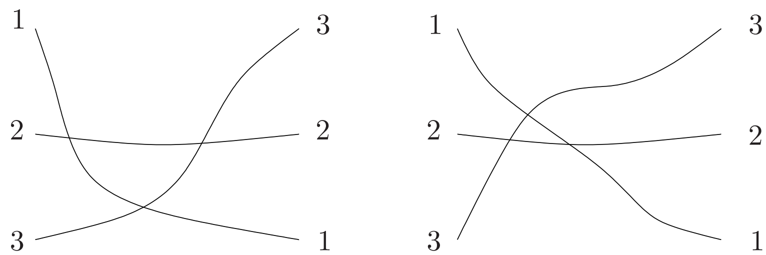

Figure 6: These arrangements are combinatorially equivalent, but not isomorphic by local sequences.

Let us call a simple Euclidean (≥ 5)-gon–free arrangement on n pseudolines with exactly k pairs of twin pseudolines a (k, n)*–arrangement*.

A pseudoline P in arrangement $\mathcal{A}$ is a *twin pseudoline in* $\mathcal{A}$ if there is a pseudoline Q in $\mathcal{A}$ such that P and Q are twin pseudolines in $\mathcal{A}$. If no such pseudoline Q exists, then P is *nontwin in* $\mathcal{A}$.

For integers $k \geq 1, n \geq 4$, let $A(k, n)$ denote the number of nonisomorphic (k, n)–arrangements (thus, in particular, it follows from Theorem 1 that $A(1, n) = 0$ for every $n \geq 4$). Consider a (k, n)–arrangement $\mathcal{A}$, and a twin pseudoline P in $\mathcal{A}$. If we add a pseudoline Q to $\mathcal{A}$, so that P and Q are twin in $\mathcal{A} \cup \{Q\}$, the result is a $(k, n+1)$–arrangement. Now there are two (nonisomorphic) ways to add such a pseudoline Q. Since there are $2k$ twin pseudolines in $\mathcal{A}$, it follows that there are $4k$ such ways to generate a $(k, n+1)$–arrangement from $\mathcal{A}$. Now consider a $(k-1, n)$–arrangement $\mathcal{B}$, and a nontwin pseudoline P in $\mathcal{B}$. If we add a pseudoline Q to $\mathcal{B}$, so that P and Q are twin in $\mathcal{B} \cup \{Q\}$, the result is a $(k, n+1)$–arrangement. Since there are two (nonisomorphic) ways to add such a pseudoline Q, and there are $n - 2(k-1)$ nontwin pseudolines in $\mathcal{B}$, it follows that there are $2(n - 2(k-1))$ such ways to generate a $(k, n+1)$–arrangement from $\mathcal{B}$. It is not difficult to check that if we perform the first operation on each twin pseudoline in each (k, n)–arrangement, and the second operation on each nontwin pseudoline in each $(k-1, n)$–arrangement, the global result we obtain is that every $(k, n+1)$–arrangement gets generated exactly k times. Therefore we obtain the recurrence $4k \cdot A(k, n) + 2(n - 2(k-1)) \cdot A(k-1, n) = k \cdot A(k, n+1)$, for $k \geq 2$ and $n \geq 4$.

Using this recurrence we may obtain the exact number of nonisomorphic simple Euclidean (≥ 5)-gon–free arrangements on n pseudolines, for every n. Now to obtain an easy bound, first note that $A(2, 4) = 8$. This follows since all arrangements with 4 pseudolines have two pairs of twin pseudolines, and there are 8 nonisomorphic arrangements with 4 pseudolines (see [9] or [10]). Using the recurrence and $A(2, 4) = 8$, we obtain $A(2, n) = 2 \cdot 4^{n-3}$. Thus the number of nonisomorphic simple Euclidean (≥ 5)-gon–free arrangements with *exactly* two pairs of twin pseudolines is already exponential in n.

4.2 Combinatorial equivalence

Throughout this section we consider the criterion under which two arrangements are isomorphic (or *combinatorially equivalent*) if there is an incidence– and dimension–preserving bijection between their induced cell decompositions (see [3]). It is readily checked that, up to isomorphism, there is exactly one arrangement with 3 pseudolines, and exactly one arrangement with 4 pseudolines.

For each $n \geq 3$ let $\mathcal{T}_n$ denote the set of all rooted full binary trees on $2n-1$ vertices (consequently n leaves), in which every leaf is labeled with a 1, and the label of each internal vertex is the sum of the labels of its two children. Thus the root of each tree in $\mathcal{T}_n$ is labelled n. The key observation is that, for each $n \geq 3$, there is a set $\mathbf{A}_{n+1}$ of pairwise nonisomorphic (≥ 5)-gon–free arrangements of $n+1$ pseudolines, and a two–to–one mapping from the set $\mathcal{T}_n$ to $\mathbf{A}_{n+1}$. The correspondence is illustrated in Figure 7. The mapping is two–to–one because the arrangements obtained from a tree and its reflection (around the root) are equivalent. On the other hand, if T' is not the reflection around the root of T, then the arrangements induced by T' and T are not equivalent: this is easily checked if we view the arrangement from the perspective of the topmost horizontal line. A standard counting argument shows that $|\mathcal{T}_n|$ grows exponentially with n. Thus under this type of isomorphism there are exponentially many nonisomorphic simple Euclidean (≥ 5)-gon–free arrangements.

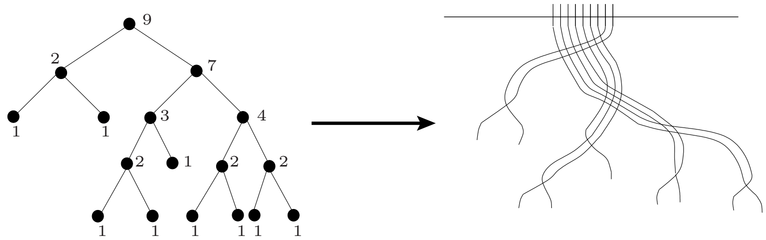

Figure 7: How to associate a (≥ 5)-gon–free arrangement with $n+1$ pseudolines to each labelled tree in $\mathcal{T}_n$. Here we show a labelled tree in $\mathcal{T}_9$ and its associated (≥ 5)-gon–free arrangement with 10 pseudolines.

Acknowledgements

We thank two anonymous referees for their valuable comments. Subsection 4.1 was not present in the original version. Its addition was inspired on a referee's thoughtful remarks.

References

[1] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, G. Ziegler. Oriented matroids. Encyclopedia of Mathematics and its Applications, **46**. Cambridge University Press, Cambridge, 1993.

[2] P. Bose, H. Everett, and S. Wismath, Properties of Arrangements. *Int. Journal of Comp. Geom.*, **13(6)** (2003), 447–462.

[3] S. Felsner, Geometric Graphs and Arrangements. Some chapters from combinatorial geometry. Advanced Lectures in Mathematics. Friedr. Vieweg & Sohn, Wiesbaden, 2004.

[4] S. Felsner, On the number of arrangements of pseudolines, *Disc. Comput. Geom.* **18** (1997), 257–267.

[5] S. Felsner and K. Kriegel, Triangles in Euclidean arrangements, *Discrete Comput. Geom.* **22** (1999), no. 3, 429–438.

[6] D. Forge and J.L. Ramírez–Alfonsín, Straight line arrangements in the real projective plane, *Discrete Comput. Geom.* **20** (1998), no. 2, 155–161.

[7] J.E. Goodman, R. Pollack, Proof of Grünbaum's conjecture on the stretchability of certain arrangements of pseudolines. *J. Combin. Theory Ser. A* **29** (1980), no. 3, 385–390.

[8] B. Grünbaum, Arrangements and Spreads. American Mathematical Society, Providence, RI, 1972.

[9] D. Knuth, Axioms and Hulls. *Lecture Notes in Computer Science* **606**. Springer–Verlag, Berlin (1992).

[10] H. Krasser, Order Types of Point Sets in the Plane. Ph.D. Thesis (2003).

[11] J. Richter–Gebert and G.M. Ziegler, Oriented matroids, in *Handbook of discrete and computational geometry*, 2nd. Ed. (J.E. Goodman and J. O'Rourke, eds.), pp. 129–151. Chapman & Hall/CRC, Boca Raton, FL, 2004.

[12] G. Ringel, Teilungen der Ebene durch Geraden oder topologische Geraden. *Math. Z.* **64** (1956), 79–102.

[13] J.P. Roudneff, The maximum number of triangles in arrangements of pseudolines. *J. Combin. Theory Ser. B* **66** (1996), no. 1, 44–74.

[14] J.P. Roudneff, Quadrilaterals and pentagons in arrangements of lines. *Geom. Dedicata* **23** (1987), no. 2, 221–227.

[15] J.P. Roudneff, On the number of triangles in simple arrangements of pseudolines in the real projective plane. *Discrete Math.* **60** (1986), 243–251.

[16] P.W. Shor, Stretchability of pseudolines is NP-hard. *Applied geometry and discrete mathematics*, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., **4**, 531–554. AMS Providence, RI, 1991.