Paths of Trains with Two-Wheeled Cars

Luis Montejano[*], and Jorge Urrutia[*†]

June 17, 2004

Abstract

In this paper we study the following simple and mind-puzzling problem: Can a model train car, which runs along an intricate track complete a full cycle around it? In our paper a track will be represented by a simple closed curve, and the cars of our model train by segments whose end-points lie on the curve.

1 Introduction

Let C be a simple closed curve in the plane that can be thought of as a track. Let us imagine a model train car with a single wheel at each end, which we run along the track of C. We ask the following question. What lengths λ may the car have that allow it to traverse all of C? If C is a circle, any car whose length λ is less than or equal to the diameter of C will be able to run around the entire length of the track. If C is an ellipse, any car with λ greater than the length of its smallest axis will, however, get stuck.

Consider a train of n such cars linked together, traveling along the curve C. Once again, it is interesting to ask what car lengths will allow the train to traverse the entire curve.

We encourage the reader to try experimenting with trains of a variety of car lengths running along different curves before proceeding to read the remainder of the article. It is an entertaining, instructive and sometimes surprising exercise to work out trajectories that will allow the train to complete

[*]Instituto de Matemáticas Universidad Naciońal Autónoma de México, México D.F. México

[†]Supported by CONACYT of Mexico, Proyecto 37540-A.

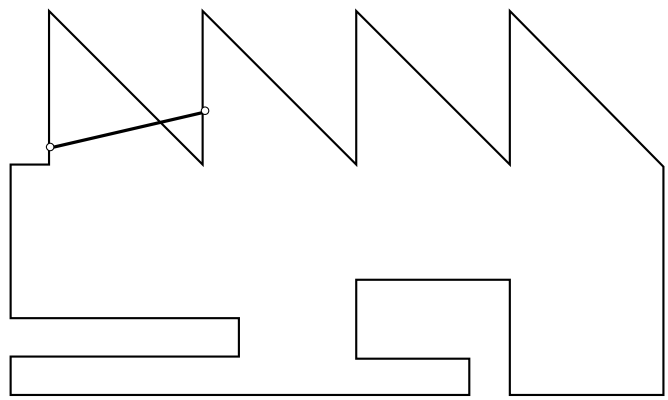

Figure 1:

a circuit of the entire curve. For example the reader may verify by himself that for the curve shown in Figure 1, the car represented by a line segments with small circles at its end-points can traverse the whole curve. In a very nice paper, Goodman, Pach and Yap [2] studied a closely related problem and obtained similar results.

Let $\alpha : S^1 \to R^2$ be a parameterization of the simple closed curve C. We shall require here that α be an injective, piecewise differentiable function. Consider the function

$$\Lambda : S^1 \times S^1 \to R$$

given by $\Lambda(x, y) = \parallel \alpha(x) - \alpha(y) \parallel$, for all $x, y \in S^1$.

$\Lambda^{-1}(\lambda)$ is the space of all possible positions of cars of length λ on curve C.

Definition. *We shall say that a car of length λ traverses the entire curve C if the following function exists:*

$$\psi = (\psi_1, \psi_2)\colon S^1 \to \mathit{\Lambda}^{-1}(\lambda) \subset S^1 \times S^1$$

such that $\psi_1 : S^1 \to S^1$ has degree ± 1, where $\psi_1, \psi_2 : S^1 \to S^1$ are the coordinate functions of ψ.

Informally speaking, we say that a car of length λ can traverse entirely curve C if its back wheel traverses C essentially once.

2 Questions About Car Paths

Question 1. Can a car of length λ traverse C without repeating a position, but in such a way that its back wheel traverses C essentially more than once?
Question 2. Let $\psi : S^1 \to \mathit{\Lambda}^{-1}(\lambda) \subset S^1 \times S^1$ be a path by which a car of length λ traverses the entire curve C, that is, a route in which the back wheel traverses C essentially once. Is it true that the front wheel then also traverses the curve once? More formally, is it true that if $\psi_1 : S^1 \to S^1$ has degree ± 1, then $\psi_2 : S^1 \to S^1$ also has degree ± 1?

One possible reason why a car longer than the minor axis of an ellipse could get stuck and be unable to traverse the entire ellipse, is if, as it turns, its orientation would become parallel to the orientation of the minor axis of the ellipse. This is not, however, possible; motivating the next question.

Question 3. If a train traverses entirely the curve C, is it true that the orientations of its cars describe a complete revolution?
Question 4. If a car of length λ traverses entirely the curve C, and $\lambda' < \lambda$, then is there a car of length λ' which can traverse C completely?
Question 5. Is traversing the entire curve C by a car of length λ a local or a global problem? In other words, is it possible for a "nice subarc" of C to exist that allows a car of length λ to traverse all of C (in which the definition of "nice" involves only the the subarc itself)?

The last question arises from a situation such as that shown in Figure 2. Furthermore, this example suggests that an affirmative answer to Question 4 is unlikely.

Our problem has aspects that make it more intriguing, as in some cases, the *starting position* of the car determines if a car car traverse all of C or not. This situation is illustrated in Figure 3.

3 Answers to Section 2

The first observation we make is that for $\lambda > 0$,

$$\Lambda^{-1}(\lambda) \cap \Delta = \emptyset$$

where $\Delta = \{(x, x) \mid x \in S^1\} = \Lambda^{-1}(0)$ is the diagonal of $S^1 \times S^1$.

A traversal of C by cars of length λ is thus a function $\phi = (\phi_1, \phi_2) : S^1 \to \Lambda^{-1}(\lambda) \subset S^1 \times S^1 - \Delta$.

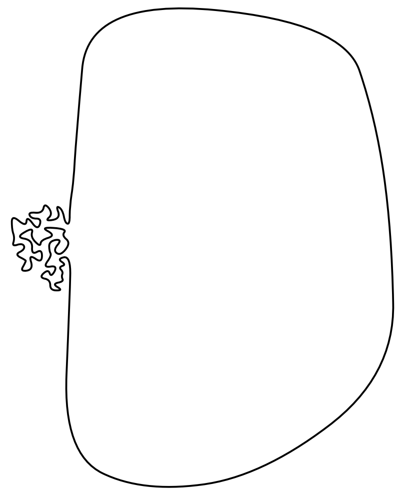

Figure 2: In this figure, C is essentially a circle in which a small portion of the circle has been replaced by a sector of a curve which can be as intricate as we might want. It is clear that a sufficiently long car can always traverse all of C. A small car might have problems getting out of the "intricate" sector of C.

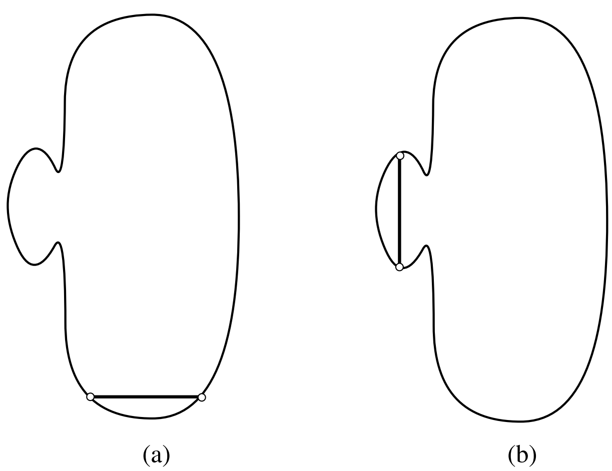

Figure 3: The reader can easily verify that a car starting as shown in (a) can traverse the whole C, whereas in (b) it is stuck!.

Functions of the circle on the torus are classified homotopically by pairs of whole numbers. That is, a function ϕ has type (n, m) if it wraps around the meridian of the torus n times and m times around its length. Two functions of the circle are homotopic on the torus if and only if they have the same type. Recall that the only curves of type (n, m) which are not self-intersecting are those which have n and m relatively prime. Moreover, if the image of the function ϕ does not intersect the diagonal Δ, then ϕ has type (n, n) for some integer $n \in Z$. More details can be found in [2].

With this in mind, a traversal of C by a car of length λ which takes on distinct positions is an injective function $\phi : S^1 \to \Lambda^{-1}(\lambda) \subset S^1 \times S^1$ of type $\pm(n, n)$ with $n = 0, 1$. If $n = 0$, the curve deforms to a point on the torus, and $\phi_1 : S^1 \to S^1$ is therefore of degree 0. If $n = 1$, then by definition, both $\phi_1 : S^1 \to S^1$ and $\phi_2 : S^1 \to S^1$ are of degree ± 1. This provides a negative answer to Question 1, an affirmative answer to Question 2, and allows us to formulate the characterization expressed in the following theorem.

Theorem 1 *The function $\psi = (\psi_1, \psi_2) : S^1 \to \Lambda^{-1}(\lambda) \subset S^1 \times S^1$ represents a car of length λ which traverses C entirely if and only if $\psi : S^1 \to \Lambda^{-1}(\lambda) \subset S^1 \times S^1$ is a function of type $\pm(1, 1)$; in other words, if and only if* $\deg(\psi_1) = \deg(\psi_2) = \pm 1$.

To answer Question 3, let α be differentiable, and define the function

$$\Theta : S^1 \times S^1 \to S^1$$

as follows: $\Theta(x, y) = \frac{\alpha(x) - \alpha(y)}{\|\alpha(x) - \alpha(y)\|}$, if $x \neq y$ and $\Theta(x, x) = \frac{\alpha'(x)}{\|\alpha'(x)\|}$.

As C is a simple closed curve, the successive directions of the tangents to C will describe a complete revolution as C is traversed completely. That is, the function $\delta\Theta : S^1 \to S^1$ has degree ± 1 where $\delta : S^1 \to S^1 \times S^1$ is the diagonal function$\delta(x) = \pm(x, x)$, for all $x \in S^1$. By Theorem 1, $\psi : S^1 \to \Lambda^{-1}(\lambda) \subset S^1 \times S^1$ represents the path of a car of length λ which traverses the entire curve if and only if ψ is homotopic to δ; therefore if and only if $\psi\Theta : S^1 \to S^1$ has degree ± 1. This gives rise to the following

Theorem 2 *A car of length λ traverses the entire smooth curve C if and only if the orientations of its cars make a complete revolution.*

We will now answer Question 4 by means of the following theorem.

Theorem 3 *If there is a car of length λ which traverses the entire curve C and $\lambda' < \lambda$ then there is a car of length λ' which traverses C entirely.*

Proof: By the hypothesis, there exists a function $\psi : S^1 \rightarrow \mathit{\Lambda}^{-1}(\lambda) \subset S^1 \times S^1$ which represents the path of a car of length λ along the entire curve. By Theorem 1, ψ is type $\pm(1,1)$, and therefore is homotopic to the diagonal function $\delta : S^1 \rightarrow S^1 \times S^1$. Let the homotopy be $H : S^1 \times I \rightarrow S^1 \times S^1$, that is $H(x,1) = \psi(x)$ y $H(x,0) = \pm(x,x)$ for all $x \in S^1$. We note that $H(S^1 \times \{1\}) \subset \Lambda^{-1}(\lambda)$ and $H(S^1 \times \{0\}) = \Lambda^{-1}(0)$. Now let $\lambda' < \lambda$ and let us consider $H^{-1}(\Lambda^{-1}(\lambda')) \subset S^1 \times I$. It is clear that $H^{-1}(\Lambda^{-1}(\lambda'))$ separates $S^1 \times \{1\}$ from $S^1 \times \{0\}$ in $S^1 \times I$. As α is a piecewise differentiable function and the function H can be chosen in such a way as to be differentiable, $H^{-1}(\Lambda^{-1}(\lambda')) \subset S^1 \times I$ is a locally connected set, and therefore $H^{-1}(\Lambda^{-1}(\lambda'))$ contains a circle in $S^1 \times I$ which separates $S^1 \times \{1\}$ from $S^1 \times \{0\}$. Let $\xi : S^1 \rightarrow S^1 \times I$ be a parameterization of this circle, and let us note that $H\xi : S^1 \rightarrow S^1 \times S^1$ is type $\pm(1,1)$ and $H\xi(S^1) \subset \Lambda^{-1}(\lambda')$. Then by Theorem 1, $H\xi : S^1 \rightarrow \Lambda^{-1}(\lambda') \subset S^1 \times S^1$ represents the path of a car of length λ along the entire curve C. ∎

To tackle Question 5, we need to define a λ subarc. Recall that we wish to define it in terms of only a portion of the curve.

Definition Let C be a smooth curve. $\Omega \subset C$ is a λ-***subarc*** of C if there is a disc D with center $0 \in C$ such that: i) $\Omega = C \cap D$, ii) $C \cap \partial D$ consists of precisely two points $\{a,b\}$, the endpoints of Ω, and iii) for all $x \in \Omega$, the normal N_x to C at x is such that $N_x \cap D \cap C = \{x\}$.

In this definition, refering to Figure 4 the hypotheses imply that a car $[b,0]$ moves along the curve C within D until it becomes $[0,a]$ such that both wheels move forward without stopping or backing up. This follows immediately from the following lemma, which says that if the back wheel is forced to back up to enable the car to keep moving forward, then the curve C is perpendicular to the car at the back wheel.

Lemma 1 *Let $\vartheta = (\vartheta_1, \vartheta_2) : (-\epsilon, \epsilon) \rightarrow \Lambda^{-1}(\lambda)$ be a smooth function such that $\frac{d\vartheta}{dt}(0) \neq 0$. If $\frac{d\alpha\vartheta_1}{dt}(0) = 0$, then the tangent to C at the point $\alpha(y)$ is perpendicular to the line passing through $\alpha(x)$ and $\alpha(y)$ where $\vartheta(0) = (x,y)$.*

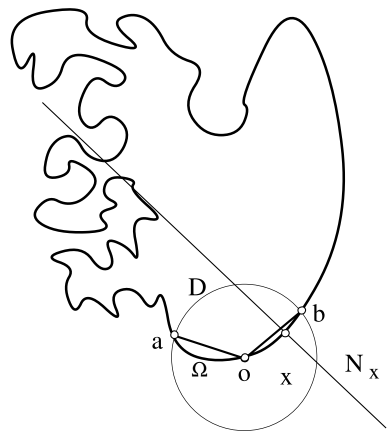

Figure 4:

Proof: It is easy to see that for $\delta > 0$ sufficiently small, we can find a smooth function $\theta : (-\delta, \delta) \to R$ such that

$$\alpha\vartheta_1(t) + \lambda(\cos\theta(t), \sin\theta(t)) = \alpha\vartheta_2(t)$$

Differentiating, and assuming that $\frac{d\vartheta}{dt}(0) \neq 0$ and $\frac{d\alpha\vartheta_1}{dt}(0) = 0$, we have that $\frac{d\alpha\vartheta_2}{dt}(0) = \frac{d\vartheta_2}{dt}(0)\frac{d\alpha}{dt}(\vartheta_2(0))$ is parallel to $(-\sin\theta(0)\cos\theta(0))$ and therefore perpendicular to the segment $\lambda(\cos\theta(0), \sin\theta(0)) = \alpha(\vartheta_2(0)) - \alpha(\vartheta_1(0))$, which implies that at the point $\alpha(x)$, the curve C is perpendicular to the line that passes through $\alpha(x)$ and $\alpha(y)$. ∎

At this point we need some elementary notions from Morse Theory and Degree Theory which we will use below. See [1] for example.

If C is a smooth curve, then the function

$$\Lambda : S^1 \times S^1 \to R$$

is a smooth function. It is easy to see that the critical points of Δ (the points $(x, y) \in S^1 \times S^1$ in which the derivative of Δ is zero) are the points of the diagonal; points in which the tangents to C at $\alpha(x)$ and $\alpha(y)$ are both perpendicular to the line through $\alpha(x)$ and $\alpha(y)$. Then $\lambda \in R$ is a critical value of Δ if $\Delta(x, y) = \lambda$ for (x, y) a critical point of Δ. If $\lambda \in R$ is not a critical value of Δ, then λ is called a regular value and $\Delta^{-1}(\lambda)$ is a finite collection of circles $\Sigma_1, ..., \Sigma_\rho$. Moreover, if the interval $[\lambda', \lambda]$ contains only regular values, then $\Delta^{-1}([\lambda', \lambda])$ is homeomorphic to the disjoint union $\Sigma_i \times [\lambda', \lambda]$, since the behavior of the function Δ changes only at the critical values.

Finally, we recall that if $f : S^1 \to S^1$ is a smooth function, then the degree of f can be calculated on a regular point x of the image in the following way. As $f^{-1}(x) = \{a_1, ..., a_\tau\}$ consists of a finite set of points, then each point a_i contributes a $+1$ or a -1 depending on whether the function f preserves or does not preserve orientation locally around a_i. The degree of f is the sum of all these $+1$'s and -1's.

The following theorem shows that the problem of traversing the entire curve with a car of length λ is not a local problem.

Theorem 4 *If C is a simple smooth curve which contains a λ-subarc, then for all $\lambda' \leq \lambda$, there is a car of length λ' which traverses the entire curve C.*

Proof: We begin by noting that λ can be assumed to be a regular value of Δ without loss of generality. Thus $\Delta^{-1}(\lambda)$ consists of a disjoint set of circles $\Sigma_1, ..., \Sigma_\rho$, each of which is in $S^1 \times S^1 - \Lambda$. That is, the type of $\Sigma_i \subset S^1 \times S^1$ is either $\pm(1, 1)$ or $(0, 0)$. If some of the circles Σ_i have type $\pm(1, 1)$, then by Theorem 1, the theorem is proved.

Consider the cars $[\alpha^{-1}(b), \alpha^{-1}(0)]$ and $[\alpha^{-1}(0), \alpha^{-1}(a)]$ in $\Delta^{-1}(\lambda)$. Suppose first that both $[\alpha^{-1}(b), \alpha^{-1}(0)]$ and $[\alpha^{-1}(0), \alpha^{-1}(a)]$ are in Σ_j, for some $j = 1, ..., \rho$. Let $\psi = (\psi_1, \psi_2) : S^1 \to \Sigma_j \subset \Delta^{-1}(\lambda) \subset S^1 \times S^1$ be a parameterization. Note that the degree of ψ_1 can be calculated using $\psi_1^{-1}(\alpha^{-1}(0)) = \{[\alpha^{-1}(0), \alpha^{-1}(b)],\ [\alpha^{-1}(0), \alpha^{-1}(a)]\}$. By Lemma 1 and the argument in the paragraph preceding the lemma, it is easy to see that the degree of ψ_1 is ± 2, which is a contradiction, as the answer to Question 1 would make this impossible.

This implies that for some $j = 1, ..., \rho$ the car $[\alpha^{-1}(0), \alpha^{-1}(a)]$ is in Σ_j but the car $[\alpha^{-1}(0), \alpha^{-1}(b)]$ is not. Let $\phi = (\phi_1, \phi_2) : S^1 \to \Sigma_j \subset \Delta^{-1}(\lambda) \subset S^1 \times S^1$ be a parameterization. We again note that the degree of ϕ_1 can be calculated using $\phi_1^{-1}(\alpha^{-1}(0)) = \{[\alpha^{-1}(0), \alpha^{-1}(a)]\}$. By Lemma 1 and the argument in the paragraph preceding the lemma, it is easy to see that the degree of ϕ_1 es ± 1, proving that there exists a car of length λ which traverses the entire curve C. The theorem now follows easily from Theorem 3. ∎

4 "Model Trains"

Definition. *A model train with n cars of lengths $\lambda_1, ..., \lambda_n$ running along the track described by C consists of $n+1$ points $\{a_1, ..., a_{n+1}\} \subset C$ such that*

i) *for $i = 1, ..., n$, $\| a_i - a_{i+1} \| = \lambda_i$*

ii) *for $i = 2, ..., n$, the points a_{i+1}, a_i, a_{i-1} orient the curve C positively.*

We also say that there exists a *model train with n cars of lengths $\lambda_1, ..., \lambda_n$ which traverses the curve C entirely* if there exists a function

$$\Psi : S^1 \to S^1 \times ... \times S^1$$

such that for all $x \in S^1$, $\{\alpha(\Psi_1(x)), ..., \alpha(\Psi_{n+1}(x))\}$ is a train running on C with n cars having lengths $\lambda_1, ..., \lambda_n$ and

$$\Psi_1 : S^1 \to S^1$$

is a function with degree ± 1, where, of course, $\Psi = (\Psi_1, ..., \Psi_{n+1})$.

We note that in this case, the projection of Ψ on the first two coordinates represents the path of a car of length λ_1 which traverses C entirely. Therefore Ψ_2, the projection of Ψ on the second coordinate, is a function of degree ± 1. Proceeding inductively, we can verify that the projection of Ψ on coordinates $i, i+1$ gives rise to the path of a car of length λ_1 which traverses C entirely; therefore $\Psi_(i+1)$ is a function of degree ± 1, $i = 0, \ldots, n$.

We will now prove the following theorem.

Theorem 5 *Suppose that a car of length λ traverses the entire curve C. Then, for any integer $n \geq 1$ and $\lambda \geq \lambda_1 \geq ... \geq \lambda_n$ there exists a train with n cars having lengths $\lambda_1, \ldots, \lambda_n$ which traverses the entire curve C.*

Proof: We begin by proving the theorem for two-car trains. By Theorems 1 and 3, let $\varphi = (\varphi_1, \varphi_2) : S^1 \to S^1 \times S^1$ be a function of type $\pm(1, 1)$ which represents the path of a car of length λ_1. Let

$$E : S^1 \times S^1 \to R$$

be a function defined by $E(x, y) = \| \alpha\varphi_2(x) - \alpha(y) \|$.

Then $E^{-1}(\lambda_2)$ is the space of all possible positions on the curve C of a two-car train with car lengths λ_1 and λ_2.

Let us consider curves $\xi_i = \{(x, \varphi_i(x)) \in S^1 \times S^1 \mid x \in S^1\}$, $i = 1, 2$. Clearly $\xi_2 \subset E^{-1}(0)$ and $\xi_1 \subset E^{-1}(\lambda_1)$ are two curves of type $\pm(1,1)$ in $S^1 \times S^1$ which do not intersect. In fact, $\Gamma = \{(x, y) \in S^1 \times S^1 \mid$ the points $\alpha\varphi_1(x), \alpha\varphi_2(x), \alpha(y)$ orient the curve C positively $\}$ is a ring in $S^1 \times S^1$ whose boundary is ξ_1 and ξ_2. If $\lambda_2 \geq \lambda_1$, then $\Gamma \cap E^{-1}(\lambda_2)$ separates ξ_1 from ξ_2 in Γ. Since α is a piecewise differentiable function and H can be chosen to be a differentiable function, $E^{-1}(\lambda_2) \subset \Gamma$ is a locally connected set and therefore contains a curve ξ_3 which separates ξ_1 from ξ_2 in Γ.

Let $\psi : S^1 \to S^1 \times S^1$ be a parameterization of ξ_3. We first note that $\psi = (\psi_1, \psi_2)$ is a curve of type $\pm(1,1)$ en $S^1 \times S^1$ with the property $\psi(S^1) \subset E^{-1}(\lambda_2) \subset \Gamma$. This implies tha the function $\Psi : S^1 \to S^1 \times S^1 \times S^1$, given by the coordinate functions $(\varphi_1\psi_{1,}\varphi_2\psi_1, \psi_2)$ represents the path of a train with two cars of lengths λ_1, λ_2 along the entire curve C.

Proceeding in the same way, it is now easy to use induction to prove the theorem true for trains with any number of cars. ■

Corollary 1 *If C is a simple smooth closed curve which contains a λ-subarc, then there exists for all $n \geq 1$ and $\lambda \geq \lambda_1 \geq ... \geq \lambda_n$ a train with n cars of lengths $\lambda_1, ..., \lambda_n$ which traverses the entire curve C.*

Proof: The proof follows immediately from Theorems 4 and 5. ■

5 Distance Traveled

Consider a polygonal P formed by two segments l_1, l_2 of lengths 3 and 4 respectively, and $2n - 1$ short and $2n$ long segments of lengths $1 + \epsilon$ and 1, and 3 and $3 + \epsilon$ respectively, as shown in Figure 5(a) for $n = 4$. Suppose that we want to to move a car R of length 1.5 (represented with the line segment with endpoints labeled b and f in Figure 5(a)) from l_1 to l_2. It is clear that to achieve our goal R must first pass trough a position in which b is on l_1 and f on the point labeled 1, Figure 5(b). Then R must move to a position in which f lies on point 2, Figure 6(a). However to achieve this, R must pass trough the positions shown in Figures 5(c), and 5(d), and then move to the position shown in Figure 6(a). A similar process has to be followed to move f to point $3, 4, \ldots$. In each of our iterations, b must move from a point on l_1 to point 1 and back to l_1. Since this has to be done n times, it follows that

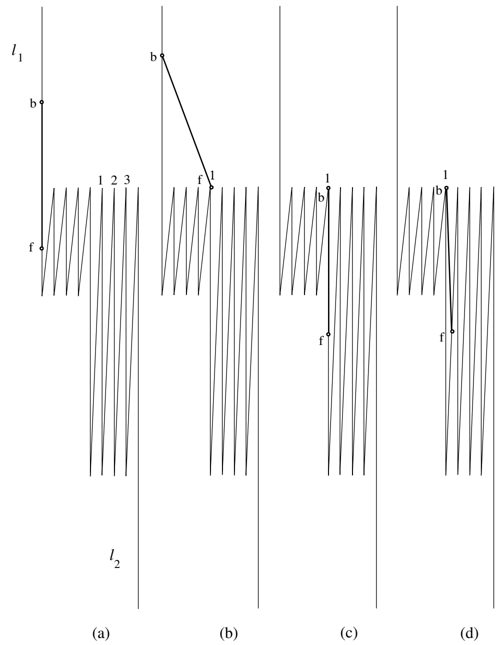

Figure 5:

the distance traveled by b, and hence f is quadratic. Since the length of P is $8n + 6 + 8\epsilon$, it follows that the distance travelled by the wheels of R can be arbitrarily large compared to the length of P.

In a similar way we can see that if instead of a car we have a train T with two cars (i.e. two segments of length 1.5 joined at one of their ends, making l_1 and l_2 longer to allow T to move), to move T from l_1 to l_2, the wheels of T must travel a distance proportional to n^3. For trains with k cars, we can easily see that the distance traveled by their wheels is $O(n^{k+1})$ (for each car we must repeat the same procedure that we did for R).

By completing P to a simple closed polygon we obtain:

Theorem 6 *Let C be a simple closed curve and let B be a car that can traverse C. Then the distance traversed by the wheels of R in a complete traversal of C can be arbitrarily large with respect to the length of C. Moreover if C is a polygon with n vertices, and has lenght $O(n)$, to complete a whole traversal of C the wheels of a train with k cars may have to travel a distance of $O(n^{k+1})$.*

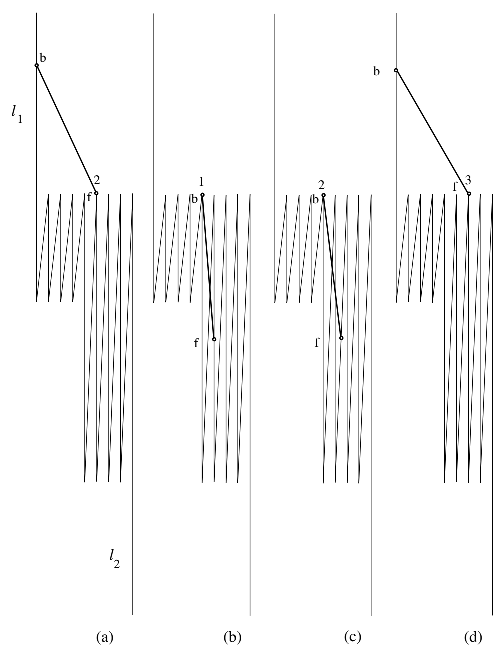

Figure 6:

Note. Except for Theorem 2, the results given here are also valid when $\alpha : S^1 \to X$ is a differentiable or piecewise differentiable (not necessarily injective) function on a Riemann manifold X. Theorem 2 is the only result in this article which depends on C being a simple curve in the plane.

References

[1] J. Milnor, *Differential Topology, Lectures on Modern Mathematics*, Vol. 2, ed. Saaty, Wiley 1964.

[2] J.E. Goodman, J. Pach, and C.K. Yap, Mountain climbing, ladder moving, and the ring-width of a polygon, *American Mathematical Monthly* Vol. 96(6), 494-510, 1989.

[3] D. Rolfsen, Knots and Links, *Mathematics Lecture Series No. 7.* Publish or Perish, Inc. 1976.