On the chromatic number of tree graphs

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Abstract

The tree graph $T(G)$ of a connected graph G has as vertices the spanning trees of G, and two trees are adjacent if one is obtained from the other by interchanging one edge. In this paper we study the chromatic number of $T(G)$ and of a related graph $T^*(G)$.

Given a connected graph G, the *tree graph* $T(G)$ is defined as the graph having as vertices the spanning trees of G, and edges joining two trees T_1 and T_2 whenever $T_2 = T_1 - e + f$ for some edges e and f.

The *adjacency tree graph* $T^*(G)$ has the same vertices as $T(G)$ but now two trees T_1 and T_2 are adjacent when $T_2 = T_1 - uv + uw$ for some adjacent edges uv and uw. Clearly $T^*(G)$ is a (spanning) subgraph of $T(G)$. See [1, 2, 3, 4] for some relevant properties of tree graphs and of some related graphs.

The aim of this note is to study the chromatic number of tree graphs and adjacency tree graphs. We give upper bounds on $\chi(T(G))$ and $\chi(T^*(G))$ in terms of basic parameters of G. We show that in some cases our bonds are optimal. For complete and complete bipartite graphs, our bounds are within a constant factor from the true values.

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Our first result is the following.

Theorem 1 *If G is a connected graph then $\chi(T(G)) \leq |E(G)|$. This bound cannot be improved in general.*

Proof. Let $m = |E(G)|$ and let $\lambda : E(G) \to \{1, 2, \ldots, m\}$ be any bijection between the edges of G and the integers from 1 to m. Then for any spanning tree T of G, define its colour as

$$c(T) = \sum_{e \in T} \lambda(e) \bmod m.$$

If $T_2 = T_1 - e + f$, then $c(T_1) - c(T_2) = \lambda(e) - \lambda(f)$, which is not 0 modulo m. Hence adjacent trees get different colours.

Finally, let G be a cycle C_n. Then $T(G) = K_n$ and in this case $\chi(T(G)) = |E(G)| = n$. □

We observe that for the case $G = K_n$ the bound given in Theorem 1 is close to optimal. Assume for simplicity that n is even. Let $V = V_1 \cup V_2$ be a bipartition of the vertex set with $|V_1| = |V_2| = n/2$, and let U_1 and U_2 be, respectively, spanning trees of V_1 and V_2. Then the family of trees $\{U_1 \cup U_2 + e\,;\ e \in E(V_1, V_2)\}$, where $E(V_1, V_2)$ is the set of edges between V_1 and V_2, is clearly a clique in $T(K_n)$ of size $n^2/4$. This gives a lower bound on the chromatic number. Combining this with Theorem 1, and taking into account the case where n is odd, we arrive at upper and lower bounds within the same order of magnitude:

$$\lfloor n^2/4 \rfloor \leq \chi(T(K_n)) \leq n(n-1)/2.$$

Observation 1. It is well-known that $\chi(H) \leq \Delta(H)+1$ for every graph H. However if $H = T(K_n)$ then $\Delta(H)$ is $O(n^3)$. This follows from the fact that the degree of a path of length n in $T(K_n)$ is equal to $(n+3)(n-1)(n-2)/6$. This is easy to show, as well as the fact that this is the maximum degree in $T(K_n)$.

Observation 2. The same argument as before shows the following: if for a given graph G there exists a bipartition $V(G) = V_1 \cup V_2$ such that the graphs induced by G on V_1 and V_2 are both connected, then $T(G)$ contains a clique of size $|E(V_1, V_2)|$. However, in general these are not the larger cliques in $T(G)$, as the example $G = C_n$ demonstrates.

Let $G = K_{n,n}$ be the complete bipartite graph with vertex partition $V = \{v_1, \dots, v_n\}$ and $W = \{w_1, \dots w_n\}$. Let $U_1 = \{v_1w_1, v_1w_2, \dots v_1w_{n-1}\}$, $U_2 = \{w_nv_2, w_nv_3, \dots w_nv_n\}$. Then the family of trees $\{U_1 \cup U_2 + e\, ; \, e \notin E(U_1) \cup E(U_2)\}$ is a clique in $T(K_{n,n})$. The size of this clique is $n^2 - 2(n-1)$ and Theorem 1 gives the exact asymptotic value of $\chi(K_{n,n})$:

$$n^2 - 2(n-1) \leq \chi(T(K_{n,n})) \leq n^2.$$

As in the case of complete graphs, it is easy to see that the maximum degree of $T(K_{n,n})$ is $O(n^3)$.

Now we proceed to give an upper bound for $\chi(T^*(G)$.

Theorem 2 *If G is a connected graph then $\chi(T^*(G)) \leq \chi'(G)$, where $\chi'(G)$ is the edge-chromatic number of G. This bound cannot be improved in general.*

Proof. Let $r = \chi'(G)$ and let $\lambda : E(G) \to \{1, 2, \dots, r\}$ be a proper edge-colouring of the edges of G. If T is a spanning tree T of G, define its colour as

$$c(T) = \sum_{e \in T} \lambda(e) \bmod r.$$

If $T_2 = T_1 - uv + uw$, then $c(T_1) - c(T_2) = \lambda(uv) - \lambda(uw)$. Since uv and uw are adjacent edges in G, they have different colour, i.e. $\lambda(uv) \neq \lambda(uw)$. Again we see that adjacent trees get different colours.

Finally, if $G = C_n$ with n even, then $T^*(G) = C_n$ and $\chi(T^*(G)) = \chi'(G) = 2$. □

Vizing's theorem says that $\chi'(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree in G. As a corollary we get that $\chi(T^*(G)) \leq \Delta(G) + 1$.

Now take $G = K_n$. It is well-known that $\chi'(K_n) = n - 1$ if n is even, and $\chi'(K_n) = n$ if n is odd. Take a path spanning $n - 1$ vertices of K_n and join the remaining vertex to this path in $n - 1$ different ways. In this way we obtain a clique in $T^*(K_n)$ of size $n - 1$. Now Theorem 1 almost gives the exact value of the chromatic number:

$$\chi(T^*(K_n)) = \begin{cases} n-1, & n \text{ even;} \\ n-1 \text{ or } n, & n \text{ odd.} \end{cases}$$

The above argument for obtaining a clique can be used to prove the following:

Theorem 3 *If G is a 2-connected graph with maximum degree Δ such that $\chi'(G) = \Delta$, then $\chi(T^*(G)) = \Delta$.*

Proof. First of all, by Theorem 2 we have $\chi(T^*(G)) \leq \Delta$. Now let v be a vertex of G of maximum degree Δ. Since $G - v$ is connected it admits a spanning tree T. Now the trees obtained by joining v to its different neighbors in T gives a clique of size Δ. □

By König's theorem this applies to bipartite graphs. In particular, if $r, s \geq 2$, we have

$$\chi(T^*(K_{r,s})) = \max(r, s).$$

To conclude, we remark that the colourings we have defined for $T(G)$ and $T^*(G)$ are simple and efficient. The colour of any tree is implicit in its set of edges, and it can be computed in $O(n)$ time, where n is the order of G.

References

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