On Polygons Enclosing Point Sets

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Abstract

Let V and W be points sets in the plane in general position with cardinality n and m, respectively. We say that a subset S of W is enclosed by V if there is a simple polygon P with vertex set V such that all the points of S are interior to P. Clearly the elements of W which are not contained in the convex hull of V cannot be enclosed by V. In this paper we prove that if W is contained in the convex hull of V, then V encloses at least half of the points of W. We also prove that if the convex hull of V has at least $\min\{56m, (2\lceil\log m\rceil + 1)m\}$ vertices, then V encloses the whole of W.

Let V be a point set in the plane with $|V| = n$. We say that a simple polygon P is a *polygonization* of V if the vertex set of P is exactly V. If the n points are in convex position, the number of possible polygonizations is just one, but this number is large in general and known to be at most c^n, for some constant $c > 0$. The extremal problem of finding the tight value of the constant c has attracted much attention and is still open [3].

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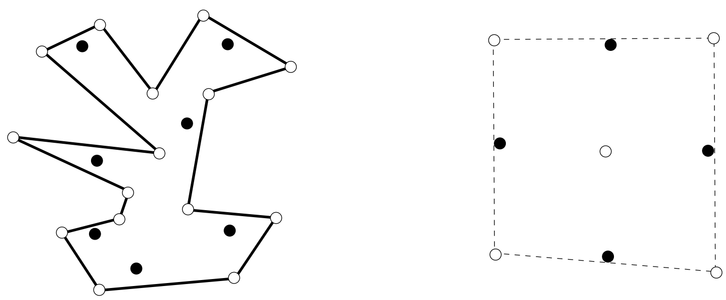
From the algorithmic viewpoint, several problems on computing the optimal polygonization according to some criterion, are computationally hard; the most popular is the *Euclidean traveling salesman problem*, which consists of finding the polygonization that minimizes the perimeter. The problems of maximizing or minimizing the area are also known to be NP-hard [2].

Let W be a second point set, with $|W| = m$ such that $V \cup W$ is in general position. We say that V *encloses* a subset S of W, or that *S is enclosed by V*, if there is a polygonization P of V such that all the points of S are interior to P (Figure 1a). Clearly the elements of W which are not contained in the convex hull of V cannot be enclosed by V. We assume hereafter that W is contained in the convex hull of V, and we will prove that in this situation V encloses at least half the points of W. We also will prove that if the convex hull of V has at least $\min\{56m, (2\lceil \log m \rceil + 1)m\}$ vertices, then V encloses the entire set W. We say that the points in V are the *vertex points* and that the points in W are the *weight points*. The weight of a polygonization of V will simply be the number of enclosed points from W.



(a) (b)

Figure 1: (a): The set of solid points is enclosed by the set of open points. (b): The set of solid points is not enclosed by the set of open points.

We first state without proof a simple lemma that will be useful later. Vertices on the boundary of a simple polygon will be ordered counterclockwise throughout the paper.

Lemma 1 *Let $e = p_1p_2$ be an edge of the convex hull $CH(V)$ of V, and $q \in W$. If no point from $V \cup W$ lies in the interior of the wedges $\widehat{p_1p_2q}$ and $\widehat{p_2p_1q}$, then any polygonization enclosing q has e as an edge.*

As a consequence, if $|CH(V)| = k < n$, a point set W not enclosed by V can be obtained by placing, for each edge e of $CH(V)$, a point q_e in the interior of $CH(V)$ at distance ϵ of the midpoint of e, for ϵ small enough (Figure1b).

We have seen in Figure1(b) that the condition of W being contained in the interior of $CH(V)$ does not guarantee that W is enclosed by V. On the other hand, computing the *maximum-weight polygonization*, i.e. the one that contains the largest number of points from W, is an NP-hard problem, as proved in [2]. Hence it is natural to ask whether at least some large subset of W would be always enclosed. Our result in this respect follows from a lemma which is intrinsically interesting on its own.

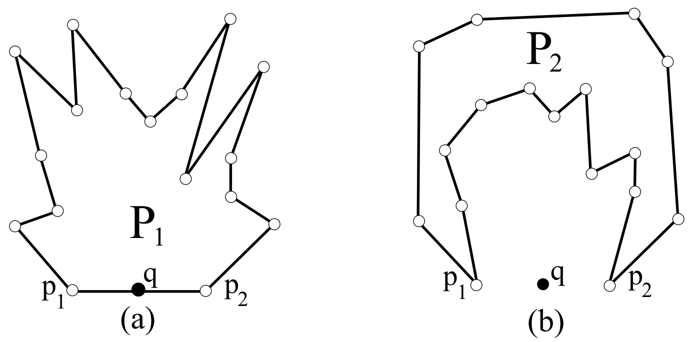


Figure 2: Two polygonizations whose union covers the convex hull.

Lemma 2 *Every point set V in general position admits two polygonizations such that their union entirely covers $CH(V)$.*

Proof: Let $e = p_1p_2$ be an edge of V, and let q be its midpoint. A first polygon P_1 is defined by connecting the points of V as they appear angularly sorted around q, plus the edge e (Figure 2a). A second polygon P_2 uses all edges in $CH(V)$ with the exception of e, and the polygonal obtained by connecting the interior points together with p_1

and p_2, as they appear angularly sorted around q (Figure 2b). It is clear that the union of P_1 and P_2 covers $CH(V)$. $\square$

Theorem 1 *Let V and W be point sets in the plane, with $|W| = m$, such that $W \subset CH(V)$ and that $V \cup W$ is in general position; then V admits some polygonization enclosing at least $\lceil m/2 \rceil$ points of W. This bound is asymptotically tight.*

Proof: The first statement is an immediate consequence of Lemma 2. To prove the second statement let us consider a point set $V = \{p_1, \ldots, p_n\}$ such that $CH(V)$ is the triangle $p_1p_2p_3$. For every point $p_i \in V$, let C_i be the circle with center p_i and radius ϵ, where ϵ is small enough for every C_i not being intersected by any line defined by two points in V, different from p_i. Place r weight points uniformly distributed on every C_i, where r is a very large number; in this way every unit angle with apex p_i will contain $r/2\pi$ weight points. Discard the weight points that lie outside $CH(V)$, i. e. suppose that W is the set of all remaining weight points. In total we have $|W| = (n-3)r+\pi(r/2\pi) = (2n-5)r/2$. Since the sum of internal angles in *any* polygonization of V is $(n-2)\pi$, it will enclose $[(n-2)\pi](r/2\pi) = (n-2)r/2$ weight points. As a fraction of the total this is $(n-2)/(2n-5)$ which for large n can be made arbitrarily close to $1/2$. $\square$

Observe that, in any polygonization P of V, the vertices of $CH(V)$ appear in P in the same order in which they appear in $CH(V)$. If $e = p_1p_2$ is an edge of $CH(V)$, we say that the vertices of P between p_1 and p_2 form the *pocket* of P with *lid* e. Next we give a lemma on pockets in a special situation.

Lemma 3 *Let p_1, p_2, p_3 be three consecutive vertices of the convex hull $CH(V)$ of V, and let q be a point interior to $CH(V)$. Then V has a polygonization P enclosing q such that all edges of $CH(V)$ are in P, except possibly p_1p_2 and p_2p_3.*

Proof: Sort all the points interior to $CH(V)$ angularly around p_2. The points in the angular interval from p_1 to q are given to the pocket with lid p_1p_2, those in the angular interval from q to p_3 are given to the pocket with lid p_2p_3 (Figure 3). $\square$

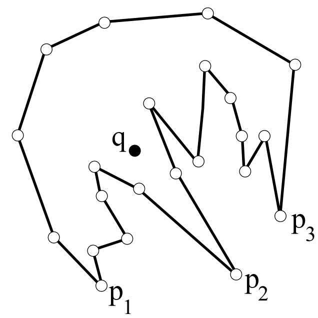


Figure 3: The construction in the proof of Lemma 3

It is natural to ask about general conditions that guarantee that a point set W will be enclosed by a point set V. We next give answers to this question related to the size of $CH(V)$.

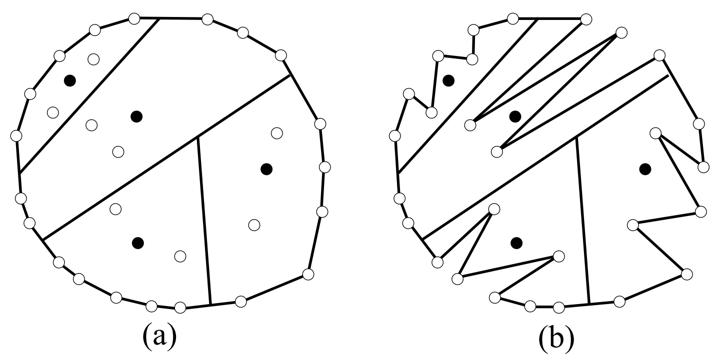


Figure 4: The split-and-polygonize process of Proposition 1

Proposition 1 *Let V and W be point sets in the plane, with $|W| = m$, such that $W \subset CH(V)$ and that $V \cup W$ is in general position. If $|CH(V)| \geq m(2\lceil \log m \rceil + 1)$ then W is enclosed by V.*

Proof: Using the discrete version of Borsuk-Ulam's theorem [1], we can simultaneously bisect W and the vertex set of $CH(V)$ with a line L. In the resulting pieces, which have L as *dividing wall*, we again bisect the set of weight points and the set of vertices from $CH(V)$, and the process is iterated until W has been split into singletons. At the end we have m pieces, and each piece K will have as boundary at most $\lceil \log m \rceil$ dividing walls and at most $\lceil \log m \rceil$ portions from the boundary of $CH(V)$. Therefore, by the pigeonhole principle, at least one of these portions will contain three consecutive vertices p_i, p_{i+1}, p_{i+2} from the vertex set of $CH(V)$. Using Lemma 3, we can construct a polygon that will use the complete boundary of K, with the possible exception of the edges $p_i p_{i+1}$ and $p_{i+1} p_{i+2}$, using as additional vertices the points from V interior to K, and enclosing the only weight point interior to K (Figure 4). Finally, we merge all these polygons by deleting the dividing walls, and obtain the desired polygonization of V enclosing W. □

According to the above proposition, 56 vertices on $CH(V)$ will suffice for enclosing $m = 7$ weight points. This observation is used in the next result, which gives a better bound for large values of m.

Proposition 2 *Let V and W be point sets in the plane, with $|W| = m$, such that $W \subset CH(V)$ and that $V \cup W$ is in general position. If $|CH(V)| \geq 56m$ then W is enclosed by V.*

Proof: The process is similar to the proof of Proposition 1. Simultaneously bisect W and the vertex set of $CH(V)$ with a line L, which will hit two boundary edges $p_i p_{i+1}$ and $p_j p_{j+1}$. Let q_i and q_j be the points $p_i p_{i+1} \cap L$ and $p_j p_{j+1} \cap L$, respectively, which we call the *dummy vertex points*. Let us consider, for example, the piece K that contains p_i and p_{j+1}. We place three new weight points, which we call *dummy weight points* very close to the midpoints of the edges $p_i q_i$, $q_i q_j$ and $q_j p_{j+1}$ (Figure 5). From Lemma 1 it follows that these three edges will be present in any polygonization of the vertex set of K (including dummy vertex points) which encloses all the weight points in K, including the dummy ones. Let A be the set of *non dummy* boundary vertices in K from $CH(V)$, and let B be the set of weight points in K *including the dummy ones*; we now repeat the process taking a line which simultaneously halves A and B.

The process is iterated until the number of weight points in each piece is less or equal than 7. The reason for this number is that we want the number of weight points in a piece to be smaller than it was

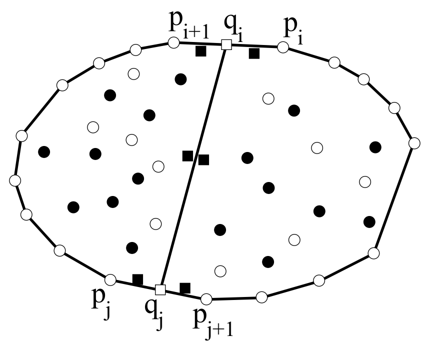


Figure 5: The process of Proposition 2. Solid squares indicate the dummy weight points and open squares indicate the dummy vertices.

before the splitting; if the number before halving is w, we have

$$\left\lceil \frac{w}{2} \right\rceil + 3 < w \iff w > 7.$$

After t splits the number of weight points in piece is at most

$$\left\lceil \frac{m}{2^t} + \frac{3}{2^{t-1}} + \frac{3}{2^{t-2}} + \cdots + \frac{3}{2} + 3 \right\rceil,$$

which becomes ≤ 7 when $t = \lceil \log m \rceil$.

The number of resulting pieces is at most m, each one containing at most 7 weight points and at least 56 vertices from $CH(V)$, due to the hypothesis and to the even split. Therefore we can polygonize the piece in such a way that the weight points are enclosed. In the last step we merge all these polygons by deleting the dividing walls, and obtain the desired polygonization of V enclosing W. □

Finally, we summarize the two propositions above into a single theorem:

Theorem 2 *Let V and W be point sets in the plane, with $|W| = m$, such that $W \subset CH(V)$ and that $V \cup W$ is in general position. If $|CH(V)| \geq min\{(2\lceil \log m \rceil + 1)m, 56m\}$ then W is enclosed by V.*

References

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