# On convex quadrangulations of point sets on the plane

V.M. Heredia[1⋆] and J. Urrutia[2⋆]

[1] Facultad de Ciencias, Universidad Nacionál Autónoma de México, México D.F. México

[2] Instituto de Matemáticas, Universidad Nacionál Autónoma de México, México D.F. México

**Abstract.** Let $P_n$ be a set of $n$ points on the plane in general position, $n \geq 4$. A *convex quadrangulation* of $P_n$ is a partitioning of the convex hull $Conv(P_n)$ of $P_n$ into a set of quadrilaterals such that their vertices are elements of $P_n$, and no element of $P_n$ lies in the interior of any quadrilateral. It is straightforward to see that if $P$ admits a quadrilaterization, its convex hull must have an even number of vertices. In [6] it was proved that if the convex hull of $P_n$ has an even number of points, then by adding at most $\frac{3n}{2}$ Steiner points in the interior of its convex hull, we can always obtain a point set that admits a convex quadrangulation. The authors also show that $\frac{n}{4}$ Steiner points are sometimes necessary. In this paper we show how to improve the upper and lower bounds of [6] to $\frac{4n}{5}+2$ and to $\frac{n}{3}$ respectively. In fact, in this paper we prove an upper bound of $n$, and with a long and unenlightening case analysis (over fifty cases!) we can improve the upper bound to $\frac{4n}{5}+2$, for details see [9].

## 1 Introduction

Let $P_n$ be a set of $n$ points on the plane in general position. A triangulation $T$ of $P_n$ is a partitioning of the convex hull $Conv(P_n)$ into a set of triangles with disjoint interiors such that the vertices of the triangles in $T$ are elements of $P_n$, and no point in $P_n$ lies in the interior of a triangle of $T$. There is an extensive literature on the study of triangulations of point sets (see e.g. [3, 8, 11, 12]); this is due to the fundamental nature of the subject, as well as to the many applications triangulations have in subjects such as mesh generation [4]. Triangulations are also highly relevant in numerous other application areas such as pattern recognition, computer graphics, solid modeling and geographic information systems.

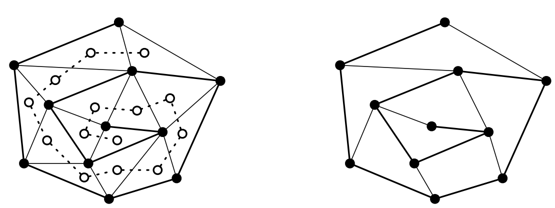
A quadrangulation $Q$ of a point set $P_n$ is a partitioning of $Conv(P_n)$ into a set of quadrilaterals with disjoint interiors such that vertices of its elements are in $P_n$, and no point in $P_n$ lies in the interior of a quadrilateral of $Q$. In general, we do not require that quadrilaterals be convex. A quadrangulation $Q$ is called *convex* if all its elements are convex. The study of quadrangulations of point sets from

⋆ Supported by CONACYT of Mexico, Proyecto SEP-2004-Co1-45876, and PAPIIT (UNAM), Proyecto IN110802.

the point of view of the present paper was started by Bose and Toussaint [5]. A good reference on quadrangulations is Toussaint's paper [14], where applications of quadrangulations to a variety of problems such as mesh generation, finite element methods, scattered data, and interpolation are mentioned.

The problem of studying quadrangulations of bicolored point sets was started recently. A point set $P_n$ is called bicolored if each of its elements is colored red or blue. A quadrangulation $Q$ of a bicolored point set is a quadrangulation of $P_n$ in which each edge of $Q$ joins a red to a blue point. In a recent paper, Cortés, Marquez, Nakamoto and Valenzuela [7] showed that a necessary (but not sufficient) condition for a bicolored point set to admit a quadrangulation is that its convex hull have an even number of points, and that consecutive points in $Conv(P_n)$ have different colors. Their main results though, concern the the flip graph of the graph of quadrangulations of bicolored point sets. Bounds on the number of Steiner points needed to quadrangulate bicolored point sets are studied in [2] and [1]. They show that by adding at most $\lceil \frac{5m}{12} + \frac{7}{2} \rceil$ Steiner colored points in the interior of a bi-colored point set, we can always obtain a new bi-colored point set that admits a bi-colored quadrangulation, and that $\lfloor \frac{4m}{12} \rfloor$ Steiner points are sometimes sufficient.

It is easy to see that a point set $P_n$ admits a quadrangulation iff the convex hull of $P_n$ has an even number of vertices. Thus from now on we will assume that *the convex hull of all point sets considered here always has an even number of vertices*. In [5], Bose and Toussaint give an elegant method to quadrangulate a point set, although the set of quadrilaterals they obtain may contain *non-convex* quadrilaterals. First they define what they call a *spiral triangulation* (see Figure 1(a)) whose dual graph contains a path from which a quadrangulation is easily obtained, see Figure 1 (a) and (b).



(a) (b)

**Fig. 1.**

In this paper we study the problem of obtaining *convex quadrangulations* of point sets. It is straightforward to see that the condition that the convex hull of $P_n$ has an even number of vertices is not sufficient to guarantee the existence of a convex quadrangulation of a point set. For example, any set $P_5$ of five points such that four of them lie on $Conv(P_5)$ admits a quadrangulation, but not a convex quadrangulation. In [6] the problem of obtaining *convex quadrangulations* of point sets by adding Steiner points is studied. Following the terminology in [6], we say that a point set $P$ can be convex-quadrangulated with at most $k$ Steiner points if by adding at most $k$ Steiner points to $P$ (located in the interior of the convex hull of $P_n$), we obtain a point set that admits a convex quadrangulation. They show that any point set can be convex-quadrangulated with at most $3\lfloor \frac{n}{2} \rfloor$ Steiner points. They also show a point set in general position for which $\frac{n}{4}$ Steiner points are necessary.

In this paper we improve on the lower and upper bounds proved in [6]. We prove that $n-1$ Steiner points are always sufficient, and that $\frac{n}{3}$ are sometimes necessary to convex-quadrangulate any point set with $n$ elements. We then outline a method for improving the upper bound to $\frac{4n}{5}+2$. This requires an extensive and unenlightening case analysis that is skipped here. Full details appear in [9].

## 2 Convex quadrangulations of point sets

In this section we prove the following result:

**Theorem 1.** *Any point set $P_n$ can be convex-quadrangulated with at most $n$ Steiner points placed in the interior of the convex hull of $P_n$.*

We now give the basic ideas of how to prove this result. The full details appear in [2]. To facilitate the presentation, we will allow the Steiner points to be placed on the boundary of $Conv(P_n)$. We then proceed to show how these points can be replaced by Steiner points in the interior of $Conv(P_n)$. We proceed as follows:

Choose the leftmost vertex on the convex hull of $P_n$, assuming without loss of generality that this point is unique, and let it be labeled $p$. Relabel the elements of $P_n - \{p\}$ by $\{p_0, \ldots, p_{n-2}\}$ in descending order according to the slope of the segments joining $p_i$ to $p$, $i = 1, \ldots, n-1$; see Figure 2(a)

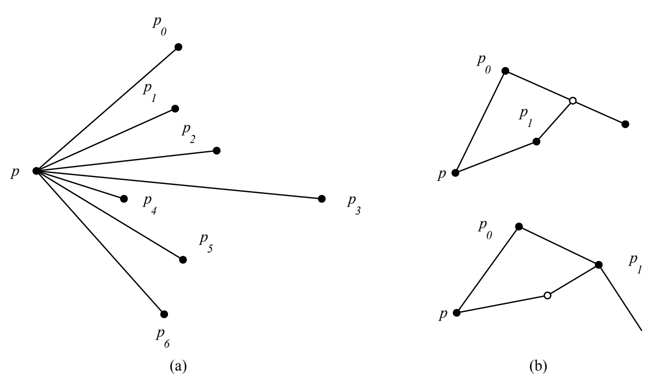



**Fig. 2.**

Consider $p_0$ and $p_1$. Two cases arise: $p_1$ lies in the interior of $Conv(P_n)$, or $p_1$ is a vertex of $Conv(P_n)$. We begin quadrangulating as shown in Figure 2(b).

Observe that in the first case, we insert a Steiner point slightly below the line segment joining $p$ to $p_1$, and in the second we place a Steiner point slightly above the line joining $p$ to $p_1$ and on the boundary of $Conv(P_n)$. We now proceed inductively, assuming that if $p_{2(i-1)+1} = p_{2i-1}$ is a vertex, there is a Steiner point slightly below the line joining $p$ to $p_{2i-1}$, or if $p_{2i-1}$ is an interior point to $Conv(P_n)$ then there is a Steiner point on the boundary of $Conv(P_n)$ slightly above the line segment joining $p$ to $p_{2i-1}$, $i \geq 0$.

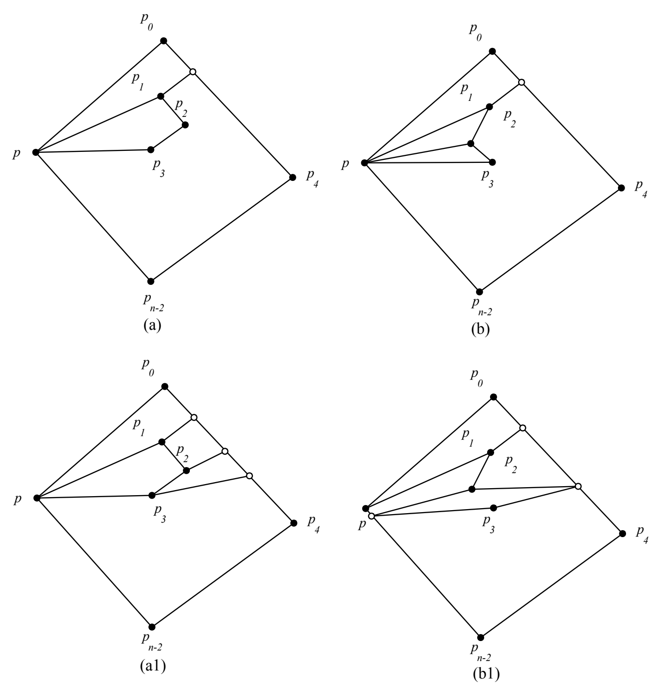



**Fig. 3.**

Assume without loss of generality that $p_{2i-1}$ is in the interior of $Conv(P_n)$. By the conditions stated in the previous paragraph, there is a Steiner point $s_j$ on the boundary of $Conv(P_n)$ slightly above the line joining $p$ to $p_{2i-1}$. Consider next $p_{2i}$ and $p_{2i+1}$, and assume that they are not vertices of $Conv(P_n)$. Two cases arise: $p_{2i+1}$ is below the line joining $p_{2i-1}$ to $p_{2i}$, or it is above it; see Figure 3(a) and 3(b), where Steiner points are shown as empty circles, and points in $P_n$ as solid small circles. The case shown in Figure 3(a) is solved as shown in Figure 3(a1).

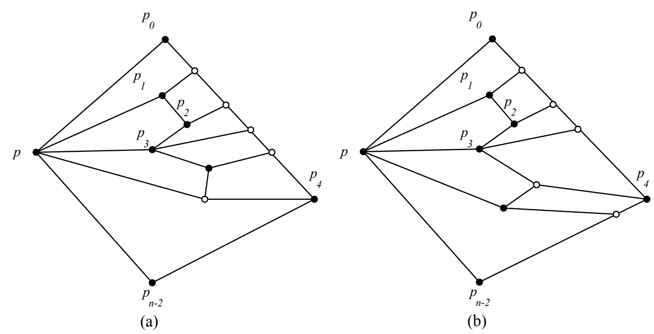


**Fig. 4.**

The case shown in Figure 3(b) requires explanation. In this case, we place a Steiner point on the edge of $Conv(P_n)$ joining $p$ to $p_{n-2}$ close enough to $p$ and relabel it $p$. A second Steiner point is placed on the boundary of $P_n$ slightly above the line that passes through $p_{2i+1}$ and the Steiner point that was relabeled $p$. We then quadrangulate as shown in Figure 3(b1). From here on. the Steiner point relabeled $p$ substitutes for $p$ in the following iterations (until it is (possibly) replaced by another Steiner point). Observe that in both cases, two points in $P_n$ were processed and two Steiner points added. The case when one of $p_{2i}$ or $p_{2i+1}$ is a vertex can be solved in a similar way; see Figure 4.

Special care should be taken when the last points to be processed are $p_{(n-2)-1}$ and $p_{2n-2}$ or when all the points of $P_n$ except $p_{2n-2}$ have already been processed. In the latter case, it may be necessary to to introduce three Steiner points. Figure 5 shows how to handle these cases. It is now easy to see that we have used at most $n$ Steiner points.

So far, we have proved that with the addition of at most $n$ Steiner points in the interior or the boundary of $Conv(P_n)$, we can convex-quadrangulate $P_n$. We now show how to modify the technique so that all the Steiner points used are located in the interior of $Conv(P_n)$. Let us assume that the edges of $Conv(P_n)$ are labeled in the clockwise direction along the boundary of $P_n$ by $e_0, \ldots, e_k$ such that $e_0$ is the edge joining $p$ to $p_0$. We observe that if an edge $e_i$ has an even number of Steiner points in it, it is straightforward to move these points to the interior of $Conv(P_n)$ and re-quadrangulate it. See Figure 6.

The problems arise when there is an edge that has an odd number of Steiner points in it, as in Figure 5(b). To solve this problem, we proceed as follows: Suppose that we have processed up to point $p_i$, and that the next pair of vertices involves a vertex of $Conv(P_n)$. Suppose that $p_{i+2}$ is a vertex of $Conv(P_n)$ (the case when $p_{i+1}$ is a vertex in $Conv(P_n)$ is handled in a similar way, and will be left to the reader). Several sub-cases arise:

1. $p_i$ and $p_{i+2}$ are vertices of $Conv(P_n)$. The normal procedure would place a single Steiner point on the edge $e_j$ joining $p_i$ to $p_{i+2}$. This case is solved

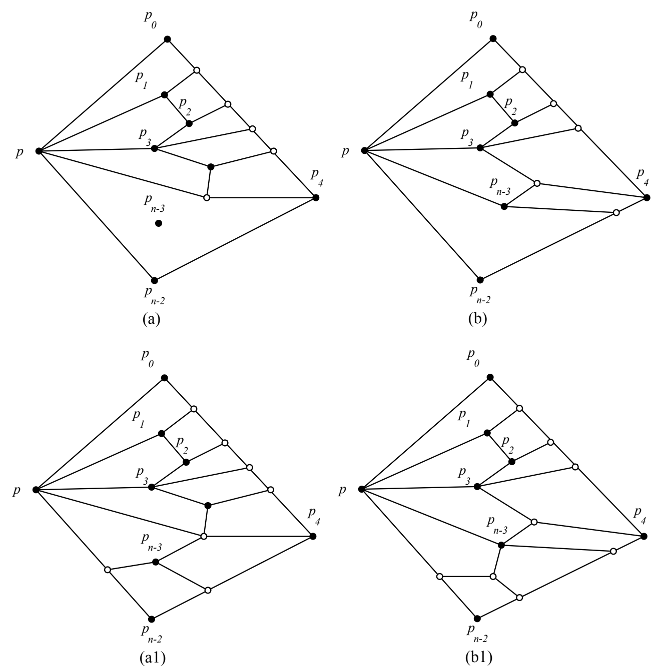



**Fig. 5.**

as shown in Figure 7(a). In this case, we can place one Steiner point in the edge starting at $p_{i+2}$ and the other on the line joining this point to $p$, and quadrangulate as shown in Figure 7(b).

2. There are an even number of Steiner points on the edge $e_j$, ending (in clockwise order) at $p_{i+2}$; see Figure 8(b). Two sub-cases arise, namely the line determined by $p_i$ and $p_{i+1}$ does not intersect $e_j$, or it does. In the first case the problem is solved as shown in Figure 8(a); in the second, as shown in Figure 8(b).

To finish the proof, we observe that once we reach $p_{n-2}$, by a cardinality argument, the edge joining $p$ to $p_{n-2}$ must have an even number of Steiner points (introduced when the point $p$ was duplicated as in Figure 3(b1) ), which by our previous observations can be moved to the interior of $Conv(P_n)$. This completes the proof of Theorem 1. □

To conclude we mention that using the same technique, but taking groups of five elements of $P_n$ instead of two, we can always convex-quadrangulate $P_n$ using four Steiner points for each five elements of $P_n$. The analysis involves studying over 50 cases, and does not give any further insight into how to improve our upper bound to what we conjecture is the correct number of Steiner points, namely $\frac{n}{2} + c$, $c$ a constant. For this reason that result is not presented here. For complete details, the reader is again referred to [9]. Thus we have:

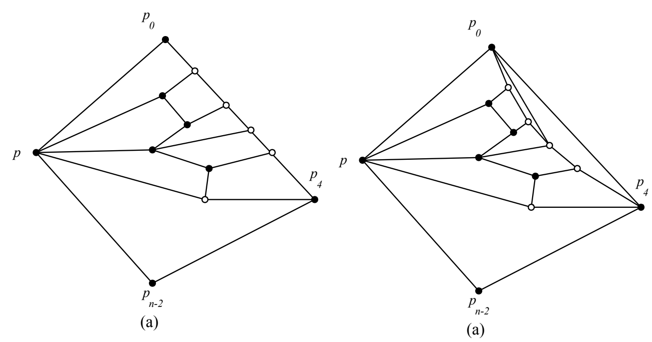



**Fig. 6.**

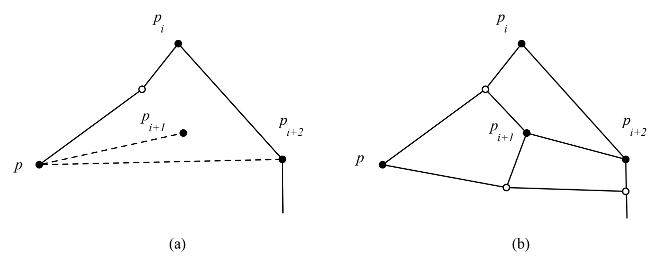



**Fig. 7.**

**Theorem 2.** *Any set $P_n$ of $n$ points can be convex-quadrangulated with at most $\lceil \frac{4n}{5} \rceil + 2$ Steiner points located in the interior of $Conv(P_n)$.*

### 2.1 Lower bounds

In [6] it was proved that there are families of point sets (not in general position) with $n$ points for which $\lceil \frac{n-3}{2} \rceil - 1$ Steiner points are needed to convex-quadrangulate them. For points in general position, an example is also presented in which $\frac{n}{4}$ points are necessary. The $\frac{n}{4}$ lower bound can be improved as follows: Consider a convex polygon $Q$ with an even number of vertices, and for every other edge of $Q$, place a point in the interior of $Q$ at distance $\epsilon$ from the middle point of the edge. An example for an octagon is shown in Figure 9(a). It is easy now to see that in any convex-quadrangulation of the point set, a Steiner point must be placed in each of the shaded polygons shown in Figure 9(b), proving that there are point sets for which $\frac{n}{3}$ Steiner points are needed to convex-quadrangulate them.

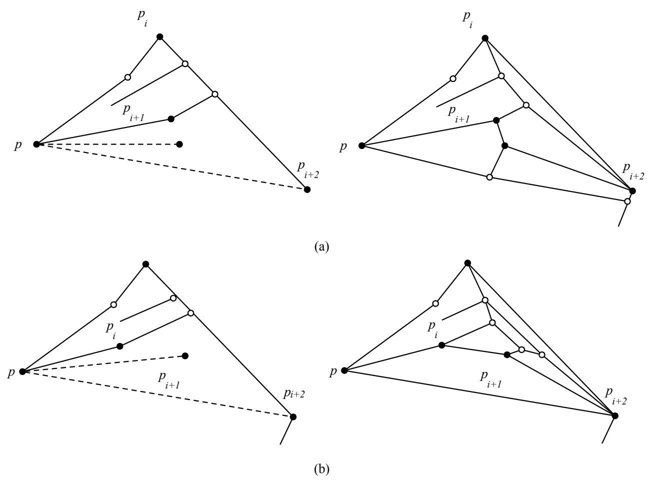



**Fig. 8.**

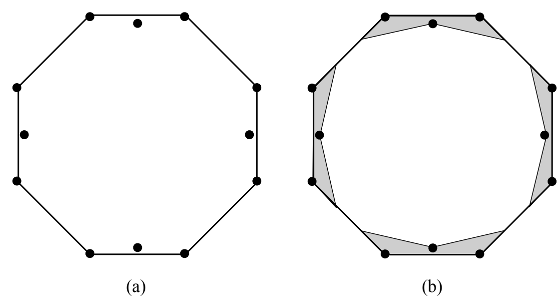



**Fig. 9.**

## 3 Conclusions

We have proved that any point set $P_n$ whose convex hull contains an even number of points can be convex-quadrangulated by adding to it at most $n$ Steiner points placed in the interior of of the convex hull of $P_n$. Using the same technique involving a long, tedious and unenlightening process, our bound can be improved to $\frac{4n}{5} + 2$ Steiner points [9]. An example where $\frac{n}{3}$ Steiner points is also presented. We believe that our upper bound and lower bounds are not tight, and that the correct values for both of them are close to $\frac{n}{2}$.

## References


1. V.M. Alvarez-Amaya, *Gráficas Geométricas Sobre Conjuntos de Puntos Coloreados.* B. Sc. Thesis, Facultad de Ciencias, Universidad Nacional Autónoma de México,

July 2006.

2. V.M. Alvarez-Amaya, T. Sakai and J. Urrutia, *Bichromatic quadrangulations with Steiner points*, in preparation.
3. F. Aurenhammer, and R. Klein, *Voronoi Diagrams*, in *Handbook of Computational Geometry*, eds. J.R. Sack and J. Urrutia, North Holland, 2000, pp. 201-290.
4. M. Bern and D. Eppstein, *Mesh Generation and Optimal Triangulation*, in *Computing in Euclidean Geometry*, D.-Z. Du and F.K. Hwang, eds., World Scientific, 1992, pp 23–90.
5. P. Bose and G. Toussaint. *Characterizing and efficiently computing quadrangulations of planar point sets.*, Computer Aided Geometric Design, vol. 14, pp 763–785, 1997.
6. D. Bremner, F. Hurtado, S. Ramaswami, and V. Sacristan, *Small strictly convex quadrilateral meshes of point sets*, Algorithmica, Volume 38, Number 2, pp 317–339, November 2003.
7. C. Cortés, A. Márquez, A. Nakamoto and J. Valenzula. *Quadrangulations and 2-colorations*. 21th European Workshop on Computational Geometry, 2005.
8. S. Fortune, Voronoi diagrams and Delaunay triangulations, in Computing in Euclidean Geometry, 2nd edition, World Scientific, 1995.
9. M. Heredia, *Cuadrilaterizaciones convexas con pocos puntos Steiner*, B. Sc. Thesis, Facultad de Ciencias, Universidad Nacional Autónoma de México, July 2006.
10. M. Heredia and J. Urrutia, *Convex quadrangulations with Steiner points*, in preparation.
11. J.M. Keil, *Polygon decomposition*, in *Handbook of Computational Geometry*, eds. J.R. Sack and J. Urrutia, North Holland, 2000, pp 491–518.
12. F. Preparata, and I. Shamos, *Computational Geometry, an Introduction*. Springer Verlag, 1985.
13. S. Ramaswami, P. Ramos and G. Toussaint. *Converting triangulations to quadrangulations.*, Computational Geometry: Theory And Applications, vol. 9, pp 257–276, 1998.
14. G. Toussaint. *Quadrangulations of planar sets*. In Proceedings of the 4th International Workshop on Algorithms and Data Structures, pages 218–227. Springer-Verlag, 1995.