On a Triangle with the Maximum Area in a Planar Point Set

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Abstract. For a planar point set P in general position, we study the ratio between the maximum area of an empty triangle with vertices in P and the area of the convex hull of P.

1 Introduction

Let P_n be a point set with n elements in general position in the plane, $n \geq 3$. For $Q \subseteq P_n$ denote the area of the convex hull of Q by $A(Q)$. We evaluate the ratio between the maximum area of an empty triangle T with vertices in P_n and the whole area $A(P_n)$. Namely, let

$$f(P_n) = \max_{T \subset P_n} \frac{A(T)}{A(P_n)}$$

and define $f(n)$ as the minimum value of $f(P_n)$ over all point sets P_n in general position. The next result proved in [5] will be used in the proof of Theorem 1.

Theorem A. Let B be a compact convex body in the plane and B_k be a largest area k-gon inscribed in B. Then $area(B_k) \geq area(B)\frac{k}{2\pi}\sin\frac{2\pi}{k}$, where equality holds if and only if B is an ellipse.

For point sets P_n in convex position (that is when the elements of P_n are the vertices of a convex polygon) the value $f^{\text{conv}}(n)$ is defined in a similar way. The following lemmas are proved in [1].

Lemma A. For point sets in convex position with five elements

$$f^{\text{conv}}(5) = \frac{1}{\sqrt{5}}.$$

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Lemma B. For point sets in convex position with six elements

$$f^{\text{conv}}(6) = \frac{4}{9}.$$

2 Points in convex position

We first study the value $f^{\text{conv}}(n)$. In what follows, a triangle with vertices x, y, z will be denoted by $\triangle xyz$.

Lemma 1. *Let $r(n)$ be the value of $f(P_n)$, where P_n denotes the set of vertices of a regular n-gon. Then*

$$r(n) = \frac{3\sqrt{3}}{2n \sin \frac{2\pi}{n}} \quad \text{when} \quad n \equiv 0 \mod 3;$$

$$r(n) = \frac{2}{n} \cdot \frac{\sin \frac{\lfloor n/3 \rfloor 2\pi}{n}}{\sin \frac{2\pi}{n}} \left(1 - \cos \frac{\lfloor n/3 \rfloor 2\pi}{n}\right) \quad \text{when} \quad n \equiv 1 \mod 3;$$

$$r(n) = \frac{2}{n} \cdot \frac{\sin \frac{\lceil n/3 \rceil 2\pi}{n}}{\sin \frac{2\pi}{n}} \left(1 - \cos \frac{\lceil n/3 \rceil 2\pi}{n}\right) \quad \text{when} \quad n \equiv 2 \mod 3.$$

Proof. Suppose that the maximum area triangle ABC with vertices in P_n divides the boundary of the convex hull of P_n into three chains AB, BC and CA, with p, q and r edges, respectively (Fig. 1, left). We show first that any two of these numbers differ at most by 1. Suppose that this is not the case, and that for r and q we have $r - q \geq 2$. Consider the $\triangle ABD$ where D is the point symmetric to C with respect to the bisector of AB (Fig. 1, right). Assume w.l.o.g. that the line AB is horizontal. Observe that $\triangle ABC$ and $\triangle ABD$ have the same area, therefore the line DC is parallel to the line AB. Observe that since $r - q \geq 2$, there is some vertex E of P_n in the arc CD, strictly above CD. Then the area of $\triangle ABE$ is greater than the area of ABC, contradicting the choice of $\triangle ABC$.

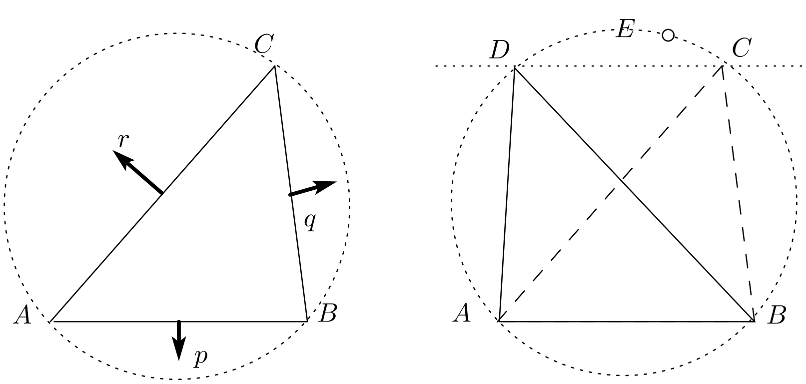

Fig.1.

Therefore, we conclude that the maximal area triangle splits the boundary into three chains whose numbers of edges are $\{t,t,t\}$, $\{t,t,t+1\}$, $\{t,t+1,t+1\}$, when $n \equiv 0,1,2 \mod 3$, respectively. An easy computation now leads to the claimed formulas. □

Notice that each $r(n)$ is a decreasing function. Thus we can deduce that

$$\lim_{n\to\infty} r(n) = \frac{3\sqrt{3}}{4\pi}\,.$$

By using Theorem A and Lemma B, we obtain:

Theorem 1. *For convex point sets in the plane of size $n > 6$ we have*

$$\frac{3\sqrt{3}}{4\pi} \leq f^{\mathrm{conv}}(9) \leq \frac{4}{9} \quad \text{and} \quad \frac{3\sqrt{3}}{4\pi} \leq f^{\mathrm{conv}}(n) \leq r(n)\ \forall n > 6, n \neq 9.$$

3 Points in general position

In this section, we estimate the value $f(n)$ for point sets in general position. A k-hole of a point set P_n is a subset $S \subset P_n$ with k elements such that the interior of the convex hull of S does not contain any element of P_n. To prove our results, we recall the well-known theorem of Harborth [2].

Theorem B. Any planar point set with 10 or more elements has a 5-hole.

We first prove the following lemma which will be useful to determine the lower bound of $f(n)$. A k-hole is said to be non-overlapping with another l-hole if these convex hulls have disjoint interiors.

Lemma 2. *Any planar point set P_{25} with 25 elements has non-overlapping three 5-holes, or one 5-hole and one 6-hole.*

Proof. Let a, b, c be three points on the plane. Let $C(a; b, c)$ denote the convex cone with apex a determined by a, b, c. We label the elements of P_{25} from p_1 to p_{25} as follows: Let p_1 be the element of P_{25} with the smallest x-coordinate. Label the remaining points $p_2, \ldots, p_{25}$ such that the slope of the line segment joining p_i to p_1 is smaller than that of the line joining p_j iff $i < j$; $1 < i, j$.

If $C(p_1; p_2, p_{17})$ contains two non-overlapping 5-holes, we are done since $C(p_1; p_{17}, p_{25})$ has one 5-hole by Theorem B. Assume otherwise. By Theorem B each of $C(p_1; p_2, p_{10})$ and $C(p_1; p_9, p_{17})$ contains a 5-hole, call them H_1 and H_2 respectively. It follows that p_{10} is a vertex of H_1 and p_9 is a vertex of H_2. Two cases arise. H_1 contains p_1 or it does not. In the first case let $\{p_1, a, b, c, p_{10}\}$ be the vertices of H_1 (labeled in the anti-clokcwise order), and consider the domain $D = C(p_1; p_{10}, p_{17}) \cap C(c; p_1, p_{10})$. If $D \cap P_{25}$ were empty H_1 would not overlap H_2. Then we can find a point u in D such that $\triangle p_1 p_{10} u$ is empty. Therefore there exist a 6-hole in $C(p_1; p_2, p_{17})$.

Suppose then that H_1 does not contain p_1 and label the vertices of H_1 $\{p_{10}, a, b, c, d\}$ in the anti-clockwise order. Let L be the line trough p_1 and p_{10}.

Rotate L in the anti-clockwise direction around p_{10} until it meets a point q in $\{p_2, \ldots, p_{25}\}$. If q is a or an interior point of $C(p_{10}; p_1, a)$, the closed half-plane determined by the line through q and p_{10} containing p_1 has precisely 18 points and we can find two convex cones whose apex is q, each containing 7 interior points. Each contains a 5-hole. Thus P_{25} contains three non-overlapping 5-holes. Suppose then that L meets a point $q \in \{p_{11}, \ldots, p_{25}\}$. Note that $C(p_{10}; p_1, q)$ contains precisely 14 interior points. Consider the point q' such that $C(p_{10}; q, q')$ contains 7 interior points. Then if q' is contained in $C(a; p_1, p_{10})$, we are done since both $C(p_{10}; q, q')$ and $C(p_{10}; q', a)$ contain at least 7 interior points each. Otherwise, let w be the point in $C(p_{10}; q, q')$ such that $C(p_{10}; w, q')$ is empty. Then both $C(p_{10}; d, w)$ and $C(p_{10}; w, p_1)$ also contain at least 7 interior points. □

Now we can prove:

Theorem 2. *Let $n \geq 25$ be an integer. Then:*

$$\frac{23}{(37+3\sqrt{5})n - (97+6\sqrt{5})} \leq f(n) \leq \frac{1}{n-1}.$$

Proof. For point sets in convex position our lower bound holds trivially. Assume that P_n is not in convex position and $A(P_n) = 1$. Assume w.l.o.g. that p_1 is in the interior of the convex hull of P_n and that p_1 is the origin. Relabel the elements of $P_n - \{p_1\}$ by $p_2, \ldots, p_n$ such that for $1 < i < j$ the angle formed by the vector p_i with the x axis is smaller than that formed by p_j with the x axis. Consider the subsets $S_k = \{p_1, p_{2+23k}, p_{3+23k}, \ldots, p_{25+23k}\}$ of P, $k = 0, \ldots, \lfloor \frac{n-2}{23} \rfloor - 1$. By Lemma 2, each S_k has three non-overlapping 5-holes, or one 5-hole and one 6-hole.

Let

$$l(n) = \frac{23}{(37+3\sqrt{5})n - (97+6\sqrt{5})}$$

and assume first that each S_k has three 5-holes. If any of these 5-holes has area greater than or equal to $l(n)\sqrt{5}$, by using Lemma A, we are done.

Triangulate each of these $3\lfloor \frac{n-2}{23} \rfloor$ 5-holes, and take a triangulation T of P_n that uses these triangles. Observe that T has at most $M(n) = (2n-5) - 9\lfloor \frac{n-2}{23} \rfloor$ triangles not contained in any of the $3\lfloor \frac{n-2}{23} \rfloor$ 5-holes of P_n.

Since each 5-hole of P_n has area smaller than $l(n)\sqrt{5}$ the total area of such 5-holes is at most $3\lfloor \frac{n-2}{23} \rfloor l(n)\sqrt{5} = L(n)$. Then at least one of the $M(n)$ triangles of T not contained in the 5-holes of P_n has area greater than or equal to

$$\frac{1 - L(n)}{M(n)} \leq l(n).$$

We claim that the value obtained when some subsets S_k of P_n have both a 5-hole and a 6-hole is larger than this lower bound. For instance, we can show by using Lemma B that the bound $l(n)$ obtained for the case in which each S_k has a 5-hole and a 6-hole is $\frac{92}{(165+4\sqrt{5})n-(422+8\sqrt{5})}$.

To prove the upper bound we construct the following configuration of n points. Take an equilateral triangle with vertices $\{u, v, w\}$ of area 1 and take a point x in this triangle such that the triangles with vertices $\{u, v, x\}$ and $\{w, x, v\}$ have area $\frac{1}{n-1}$. We place now $n-4$ points $p_5, \ldots, p_n$ on the line segment xw so that they divide the line segment xw in $n-3$ intervals of the same length. Note that each triangle with vertices $\{u, p_i, p_{i+1}\}$ has area $\frac{1}{n-1}$, $i = 4, \ldots, n$ where $p_4 = x$ and $p_{n+1} = w$. Next, move slightly $p_5, \ldots, p_n$ so that $\{v, p_4, \ldots, p_{n+1}\}$ are in convex position as shown in Fig.2. Then there is no empty triangle of area greater than $\frac{1}{n-1}$ in this configuration.

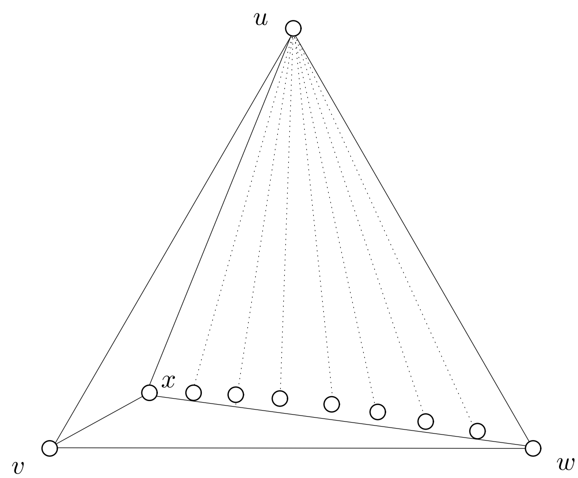

Fig.2. The configuration to realize the upper bound.

□

Notes.

1. If we define $f_d(n)$ in a similar way for d-dimensional Euclidean space, we can prove that

$$\frac{1}{dn - d^2 - d + 1} \leq f_d(n) \leq \frac{1}{(d-1)n - d^2 + 3}.$$

2. The problem studied in this paper is somehow related to the famous Heilbronn triangle problem: to place n points in a square of unit area so as to maximize the area of the smallest triangle determined by the n points. It has been proved that there is always a triangle of area $O(1/(n^{8/7-\epsilon})$ and that there are point sets in which every triangle has area $\Omega(\log n/n^2)$ ([3],[4]). It has been conjectured that the later value should be the correct one. Would the number of "small" triangles proven to be very large, one might expect to find as a consequence some "large" empty triangle, yet we have not seen so far whether this approach is feasible.

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