On the Intersection Number of Matchings and Minimum Weight Perfect Matchings of Multicolored Point Sets

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Abstract

Let P and Q be disjoint point sets with $2k$ and $2l$ elements respectively, and M_1 and M_2 be their minimum weight perfect matchings (with respect to edge lengths). We prove that the edges of M_1 and M_2 intersect at most $|M_1| + |M_2| - 1$ times. This bound is tight. We also prove that P and Q have perfect matchings (not necessarily of minimum weight) such that their edges intersect at most $min\{r, s\}$ times. This bound is also sharp. Our result is motivated by the study of the following problem: Let $P_1, ..., P_k$ be a collection of disjoint point sets in $\Re^2$ in general position, each with an even number of points. Find for each $1 \leq i \leq k$ a perfect matching M_k of P_i such that the edges of $M_1, ..., M_k$ intersect few times. We give sharp bounds for this problem too. A natural way to attack this problem is to find, for every pair i a canonical matchings M_i (e.g. minimum weight matchings), such that for every pair $\{i, j\}$ the edges of M_i andM_j have few intersections.

1 Introduction

A geometric graph is a graph whose vertices are points on the plane, and its edges line segments joining pairs of vertices. The study of geometric graphs has received considerable attention lately, see [1, 2, 3, 4, 5, 6, 8, 9, 10]. They also play an important role in Computational Geometry and many of its applications. In several of the papers cited before, problems investigating geometric graphs on multicolored point sets have been studied. Tokunaga [11]

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solved the following problem: Given two sets of points R and B in $\Re^2$ find two geometric spanning trees T_R and T_B such that the number of times their edges intersect is minimized. Kano, Merino and Urrutia [7] extended Tokunaga's result to multicolored point sets. They proved that for any collection of k point sets $P_1, ..., P_k$ there is for each i a geometric spanning tree T_i of P_i such that the edges of $T_i, ..., T_k$ intersect at most $(k-1)(n-k) + \frac{(k)(k-1)}{2}$ where $|P_1 \cup ... \cup P_k| = n$. They also prove that given two point sets P and Q their minimum weight spanning trees intersect a linear number of times, where the weight of an edge is its length.

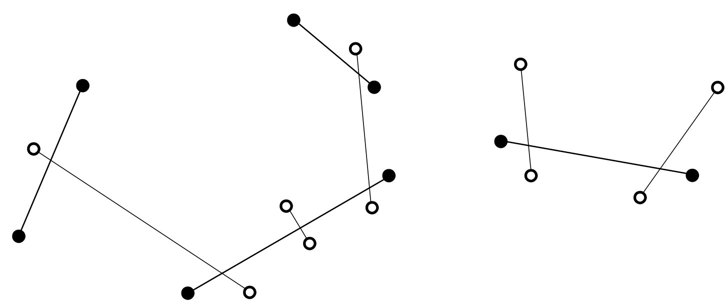

Figure 1: Minimum weight perfect matchings intersect few times

In this paper we continue our study of finding geometric graphs for multicolored point sets with few intersections. Here we study the following: Given two point sets P and Q, with $2r$ and $2s$ points respectively, find perfect matchings for each of them such that their edges intersect as few times as possible. We prove: The edges of the minimum weight perfect matchings of P and Q intersect at most $r+s-1$ times, this bound is tight. See Figure 1. As a corollary of this we obtain the following result: Let $P_1, ..., P_k$ be a colection of k disjoint point sets in $\Re^2$ such that P_i has $2r_i$ elements. Then their minimum weight spanning trees intersect at most

$$\Sigma_{i \neq j; i,j \in \{1,...,k\}} r_i + r_j - 1$$

times. This bound is tight. We also consider the problem of finding perfect matchings (not necessarily minimum weight perfect matchings) with few intersections. We prove that P and Q always have perfect matchings such that their edges intersect at most $min\{r, s\}$ times. All point sets considered here, or union of point sets considered here will be assumed to be in general position. Similarly the term graph will always refer to geometric graphs. Thus in the sequel the terms general position and geometric graphs will be omitted.

2 Minimum Weight Perfect Matchings

Let $P = \{p_1, ..., p_{2n}\}$ be a point set with $2n$ elements. A perfect matching M of P is a partitioning of P into n pairs of vertices (called the edges of M) such that every $p_i \in P$ belongs to exactly one pair of M. The weight of an edge $\{p_i, p_j\}$ of M is the length of the line segment $p_i - p_j$ joining p_i to p_j. A minimum weight matching for P is a perfect matching such that the sum of the length of its edges is minimized. Our main objective in this section is to prove the following result:

Theorem 1 *Let M and M' respectively be minimum weight perfect matchings of two disjoint point sets P and Q with $2r$ and $2s$ elements respectively. Then the edges of M and M' intersect at most $r+s-1$ times, this bound is tight.*

That our bound is tight follows from examples as shown in Figure 2.

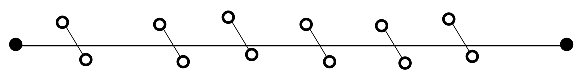

Figure 2:

Some preliminary results will be needed.

The next observation will be useful. Given two points p and q, $|p-q|$ denotes the length of the line segment joining them.

Observation 1 *Let p_1, p_2 and p_3 be three points in $\Re^2$. Suppose that for some values W and W' we have that*

$$W + |p_1 - p_2| \leq W' + |p_2 - p_3|$$

Then if we choose any point p'_2 in the interior of the segment joining p_1 to p_2 we have that

$$W + |p_1 - p'_2| < W' + |p'_2 - p_3|.$$

Our observation is clearly true when $|p_1 - p_2| = |p_2 - p_3|$, in this case $|p_1 - p'_2| < |p'_2 - p_3|$. When $|p_1 - p_2| = |p_2 - p_3| + c$, $c > 0$ let p'_1 be the point in the interior of $p_1 - p_2$ such that $|p'_1 - p_2| = |p_2, p_3|$. If p'_2 lies in the segment $p'_1 - p_2$ our result follows with $W" + |p'_1 - p_2|$ and $W = |p_2 - p_3|$. The remaining cases can be handled in a similar way. See Figure 3.

Let P be a point set with $2n$ points. Two results follow right away from our observation:

Lemma 1 *Let M be a minimum weight perfect matching of P. Suppose that the edges of M are $\{p_1, p_2\}, \{p_3, p_4\}, ..., \{p_{2n-1}, p_{2n}\}$. Then if for $i = 1, 3, ..., 2n-1$ we choose two points p'_i and p'_{i+1} in $p_i - p_{i+1}$, p'_i closer to p_i than p'_{i+1} then:*

$$|p'_1 - p'_2| + |p'_3 - p'_4| + ... + |p'_{2n-1} - p'_{2n}| < |p'_2 - p'_3| + |p'_4 - p'_5| + ... + |p'_{2n} - p'_1|$$

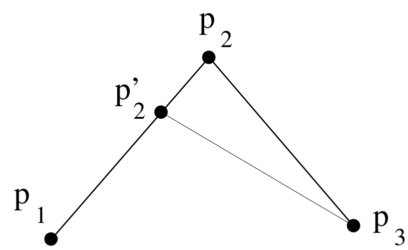

Figure 3:

Proof: Since M is a minimum weight perfect matching of P, we have:

$$|p_1 - p_2| + |p_3 - p_4| + ... + |p_{2n-1} - p_{2n}| < |p_2 - p_3| + |p_4 - p_5| + ... + |p_{2n} - p_1|.$$

Move first p_2 to p'_2, take $W = |p_3 - p_4| + ... + |p_{2n-1} - p_{2n}|$, $W' = |p'_4 - p'_5| + ... + |p'_{2n} - p'_1|$ and apply our observation. Repeat this process until each p_i has been moved to p'_i. Our result follows ∎

We also have:

Lemma 2 *Let $\{p_i, p_j\}$ be an edge of a minimum weight perfect matching M of P. Then if we choose a point $p_j^{'}$ in the interior of the segment joining p_i to p_j, then $M' = M - \{p_i, p_j\} + \{p_i, p_j^{'}\}$ is a minimum weight perfect matching of $P^{'} = P - \{p_j\} + \{p_j^{'}\}$. (See Figure 4).*

Proof: Let $\{p_i, p_j\}$ be an edge of a minimum weight perfect matching M of P. Assume w.l.o.g. that $p_i = p_1$, $p_j = p_2$. Let $p_2^{'}$ an interior point of the segment $p_1 - p_2$, and $P' = P - p_2 + p'_2$, and assume that $\mathcal{M} = M - \{p_1, p_2\} + \{p_1, p_2^{'}\}$ is not a minimum weight matching of P'_{2n}.

Let M' be a minimum weight perfect matching of P'. Clearly $\{p_1, p_2^{'}\}$ is not in M'. Then in $\mathcal{M} \cup M'$ there is a cycle C' containing $\{p_1, p_2^{'}\}$ such that its edges alternate between M and M'. By induction, we can assume that C' covers all the elements of P'.

Suppose that when we traverse the edges of C' the points of P' appear in the order $p_1, p'_2, p_3, ..., p_{2n}$. Observe that with this labeling, the edges $\{p'_2, p_3\}, ..., \{p_{2n}, p_1\}$ belong to M'. Let C be the cycle containing all the edges of C' except $\{p_1, p_2^{'}\}$ and $\{p_2^{'}, p_3\}$ plus the edges $\{p_1, p_2\}$ and $\{p_2, p_3\}$. See Figure 5.

Since M is a minimum weight perfect matching of P, we have that

$$|p_1 - p_2| + |p_3 - p_4| + ... + |p_{2n-1} - p_{2n}| < |p_2 - p_3| + |p_4 - p_5| ... + |p_{2n} - p_1|$$

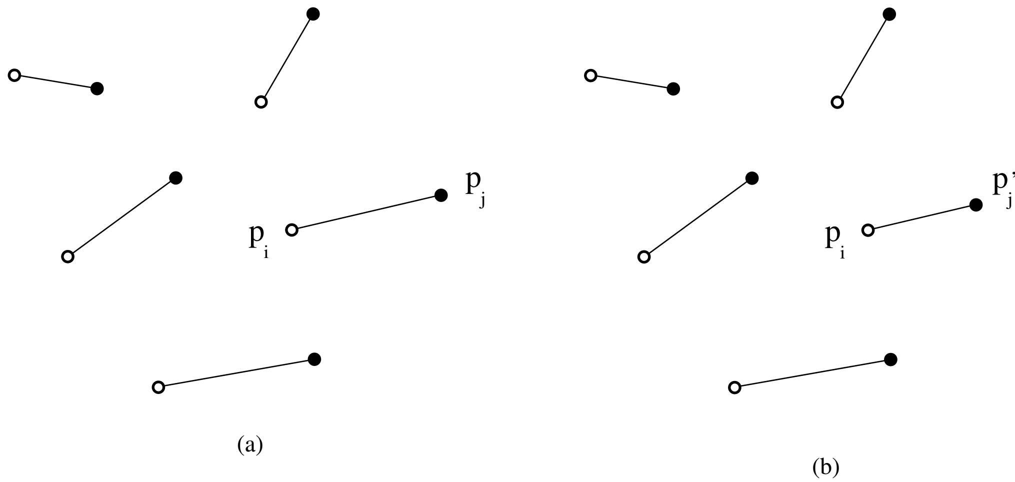

Figure 4: (a) A point set with a minimum weight perfect matching. (b) When we change p_j by a point $p_j^{'}$ on the interior of the segment joining p_i to p_j, $M - \{p_i, p_j\} + \{p_i, p_j^{'}\}$ is a minimum weight perfect matching of $P - \{p_j\} + \{p_j^{'}\}$.

If $W = |p_3 - p_4| + ... + |p_{2n-1} - p_{2n}|$ and $W' = |p_4 - p_5| ... + |p_{2n} - p_1|$, we have that

$$W + |p_1 - p_2| < W' + |p_2 - p_3|$$

But by Observation 1

$$W + |p_1 - p_2'| < W' + |p_2' - p_3|.$$

that is:

$$|p_1 - p_2'| + |p_3 - p_4| + ... + |p_{2n-1} - p_{2n}| < |p_2' - p_3| + |p_4 - p_5| ... + |p_{2n} - p_1|$$

which contradicts that M' is a minimum weight perfect matching of P'. ∎

An obvious consequence of this result is the following lemma, which we call the *Shrinking Lemma*:

Lemma 3 *Let M a minimum weight perfect matching of P. For every edge $\{p_i, p_j\}$ of M let $p_i^{'}$ and $p_j^{'}$ be two points on the* closed *line segment joining p_i to p_j. Then the set of edges $\{p_i^{'}, p_j^{'}\}$ such that $\{p_i, p_j\} \in M$ forms a minimum weight perfect matching of $P' = \{p_1^{'}, ..., p_{2n}^{'}\}$.*

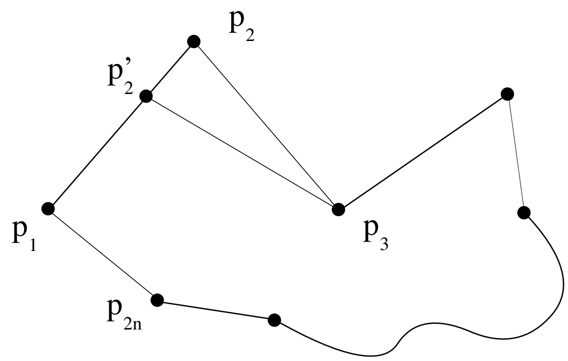

Figure 5: The thick edges belong to M.

2.1 Colored Point Sets

Let P and Q be disjoint point sets with $2r$ and $2s$ points respectively. We now prove:

Lemma 4 *Let M and M' be minimum weight perfect matchings for P and Q. Then the intersection graph of M and M' contains no cycles.*

Proof: Let G be the intersection graph of $M \cup M'$, that is the graph whose vertices are the edges of M and M', two of which $\{u, v\} \in M$, and $\{x, y\} \in M'$ are adjacent in G if the line segments $x - y$ and $u - v$ intersect. Clearly G is bipartite. Suppose now that G contains a cycle C. Then there are edges $e_1, e_3, ..., e_{2k-1}$ in M and $e'_2, ..., e'_{2k}$ in M' such that e'_i intersects e_{i-1} and e_{i+1}, $i = 2, ..., 2k-2$, and e'_{2k} intersects e_{2k-1} and e_1. Assume that the endpoints of e_i are p_i, p_{i+1}, and those of e'_i are q_{i-1}, q_i.

For $i = 2, 4, ..., 2k$ let r_{i-1} be the intersection point of e_{i-1} with e'_i. For $i = 3, 5, ..., 2k-1$ let r_{i-1} the intersection point of e'_{i-1} with e_i, and r_{2k} be the intersection point of e'_{2k} with e_1. Let C' the cycle with vertices $\{r_{2k}, r_1, ..., r_{2k-1}\}$. See Figure 6.

Assume w.l.o.g. that

$$|r_{2k} - r_1| + |r_2 - r_3| + ... + |r_{2k-2} - r_{2k-1}| > |r_1 - r_2| + |r_3 - r_4| + ... + |r_{2k-1} - r_{2k}|$$

By the Shrinking Lemma, $\{r_{2k}, r_1\}, \{r_2, r_3\}, ..., \{r_{2k-2}, r_{2k-1}\}$ is also a minimum weight perfect matching for $\{r_1, r_2, ..., r_{2k}\}$. However $\{r_1, r_2\}, \{r_3, r_4\}, ..., \{r_{2k-1}, r_{2k}\}$ is also a perfect matching for the same point set with smaller weight, which is a contradiction. Therefore G contains no cycles. By Lemma 1 the case

$$|r_{2k} - r_1| + |r_2 - r_3| + ... + |r_{2k-2} - r_{2k-1}| = |r_1 - r_2| + |r_3 - r_4| + ... + |r_{2k-1} - r_{2k}|$$

is impossible. ∎

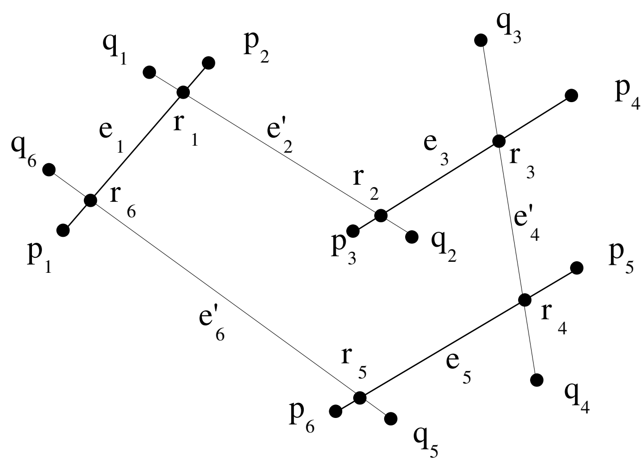

Figure 6:

Theorem 1 now follows from the fact that the intersections among the edges of M and M' are the edges of G. But since G contains no cycles and $r + s$ vertices, it contains at most $r + s - 1$ edges. ∎

2.2 Perfect Matchings

Before proceeding to study matchings for multicolored point sets, we study the problem of finding perfect matchings for bicolored point sets with few intersections removing the condition that our matchings have minimum weight. To start we recall recent results of Dumitrescu and Steiger [5]. They studied the following problem: Given two disjoint point sets P and Q find matchings M and M' for P and Q respectively, not necessarily perfect (i.e. some points in both sets may be left unmatched) such that:

1. The edges of M and M' do not intersect
2. The sum of the cardinalities of these matchings is maximized

They proved that one can always match a surprisingly high percentage of points, i.e. 83.33% of them. Their result was further improved to 85.71 % in [4] ¿From that one might get the feeling that there are always perfect matchings for P and Q such that their edges intersect cn times for c a very small constant. This unfortunately is not the case. Place on a circle $2n$ red, and $2n$ blue points such that they alternate in color. Then we can always match $2n - 2$ blue points, and $2n - 2$ red points as shown in Figure 7 without creating any intersections. However if we insist in choosing perfect matchings M and M' for our blue and red point sets, it is straightforward to prove that their edges will always intersect at least n times! To see this, simply observe that any edge joining two red points leaves an odd number of black or an odd number of red points in each of the semi-planes determined by the line containing it.

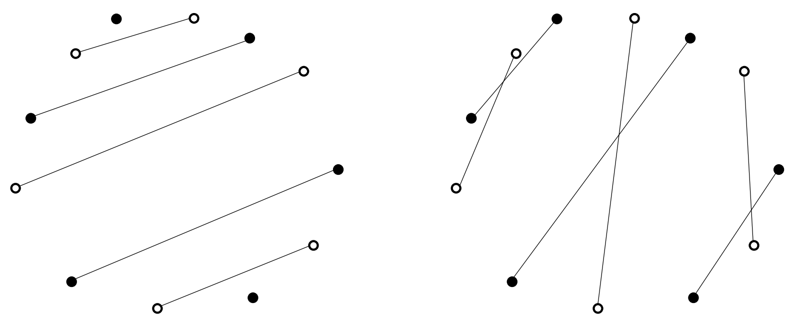

Figure 7:

We now prove:

Theorem 2 *Given any two point sets P and Q with $2r$ and $2s$ elements respectively, we can always find perfect matchings for them such that their edges intersect at most $min\{r, s\}$ times. Our bound is tight.*

Proof: Consider the convex hull $Conv(P \cup Q)$ of $P \cup Q$. If any two consecutive vertices of $Conv(P \cup Q)$ belong to P then we can match them, remove them from P, and proceed by induction. Suppose then that the vertices of $Conv(P \cup Q)$ alternate between P and Q, and that the leftmost vertex of $Conv(P \cup Q)$ belongs to P.Let us label this point p_1 and relabel the remaining points in $P - \{p_1\}$ $\{p_2, ..., p_{2r}\}$ such that if $i < j$, then p_j lies above the line joining p_1 to p_i. In a similar way label the points in Q $\{q_1, ..., q_{2s}\}$, see Figure 8. Observe that below (respectively above) the line joining p_1 to p_2 (resp. p_1 to p_{2s}) there is exactly one point in Q; If there were at least two, we could pick and match two of them such that the line segment joining them does not intersect the convex hull of the remaining elements in $P \cup Q$, and proceed by induction.

For each $1 < i < 2r$, let the wedge W_i be the region obtained by intersecting the semilpane above the line joining p_1 to p_i with the semiplane below the line joining p_1 to p_{i+1}, see Figure 8. Let k be the index such that W_k $q_2 \in W_k$. If $k > 1$ is odd, split P into two subsets, $R = \{p_1, ..., p_{k-1}\}$ and $S = \{p_k, ..., p_{2r}\}$, and Q into $R' = \{q_1, q_2\}$ and $S' = \{q_3, ..., q_{2s}\}$. Our result follows by induction on the pairs of sets R and R', and S and S'. If $k > 2$ is even then split P into $R = \{p_2, ..., p_{k-1}\}$, $S = \{p_1, p_k, ..., p_{2r}\}$, and Q into $R' = \{q_1, q_2\}$, $S' = \{q_3, ..., q_{2s}\}$, and proceed again by induction. ∎

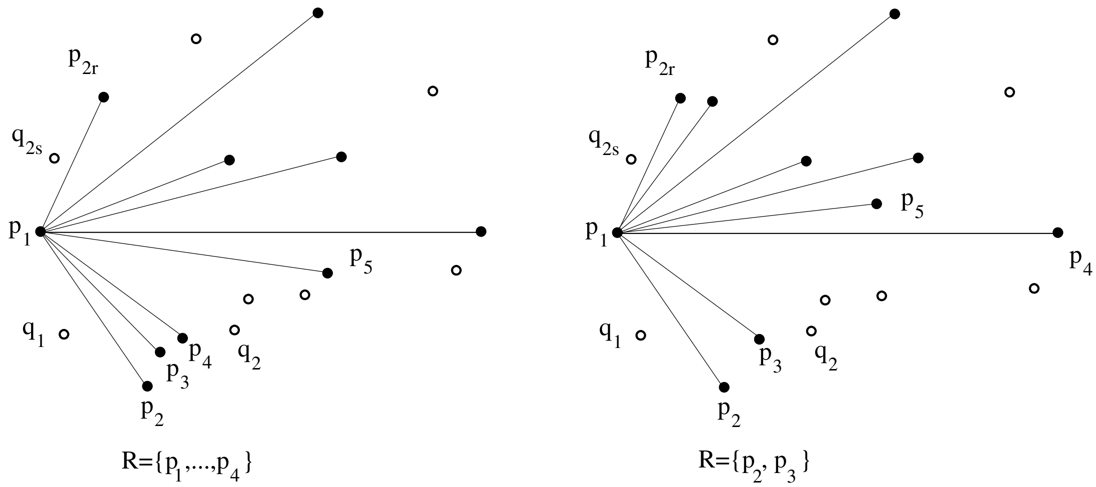

Figure 8:

2.3 Minimum Weight Perfect Matchings in Multicolored Point Sets

Consider a set of points P with $2n = 2n_1 + ... + 2n_k$ points such that for each i, $1 \leq i \leq k$, $2n_i$ elements of P are colored with color i. For each i let P_i be the set of elements of P with color i. Let M_i be a minimum weight perfect matching of P_i. We prove:

Theorem 3 *The edges of $M_1, ..., M_k$ intersect at most $(k-1)n - \frac{k(k-1)}{2}$ times. The bound is sharp.*

Proof: By Theorem 1 for every i, j the edges of M_i and M_j intersect at most $n_1 + n_j - 1$ times. Adding over all pairs $i \leq i < j \leq k$ we get our upper bound. To show that our bound is tight, we construct a point set P as before in which for every pair of matchings M_i and M_j their edges intersect exactly $n_i + n_j - 1$ times.

Consider a set R with $2n$ points in a straight line labeled $p_1, ..., p_{2n}$ from left to right, such that the distance between any two of them is at least 1. It is straightforward to see that in the minimum weight perfect matching of R, p_{2i-1} and p_{2i} are matched, $i = 1, ..., n$. If instead of a straight line we place the elements of R on an almost flat convex arc $\mathcal{C}$, i.e a convex arc contained in a rectangle of size $\epsilon \times m$, $m > 2kn$ (see Figure 9), the minimum weight perfect matching for R remains the same. Suppose now that on the same convex curve we place $4n$ points labeled $p_1, q_1, p_2, p_3, q_2, q_3, p_4, ..., p_{2n-1}, q_{2n-2}, q_{2n-1}, p_{2n}, q_{2n}$, again any two of them at distance 1, then the minimum weight perfect matchings for $R = \{p_1, ..., p_{2n}\}$ and $S = \{q_1, ..., q_{2n}\}$ intersect exactly $2n - 1$ times, see Figure 9.

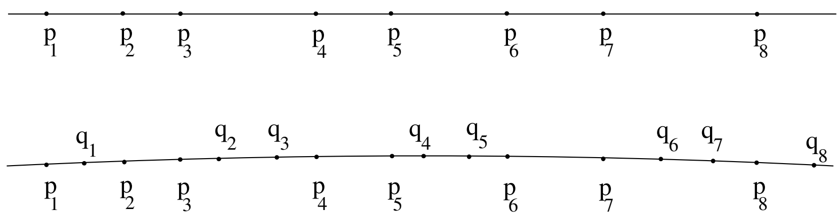

Figure 9:

Finally place $n = 2kr$ points on $\mathcal{C}$ such that any two consecutive points of our point set are at distance 1. Color them in such a way that their colors follow the next sequence of numbers:

$$1, 2, 3, ..., k, 1, 1, 2, 2, 3, 3, ..., k, k, ..., 1, 1, 2, 2, 3, 3, ..., k, k, 1, 2, 3, ..., k$$

in such a way that for every i there are exactly $2r$ points with color i. It is easy to verify that for every pair of numbers $1 \leq i < j \leq k$ the edges of the minimum weight perfect matchings of the point sets containing the points colored i and j respectively intersect $2r - 1$ times. ■

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