Upper bound constructions for untangling planar geometric graphs*

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Abstract

For every $n \in \mathbb{N}$, there is a planar graph $G = (V, E)$ with n vertices and an injective map $\pi : V \to \mathbb{R}^2$ such that in any *crossing-free* straight-line drawing of G, at most $O(n^{.4965})$ vertices $v \in V$ are at position $\pi(v)$. This improves on an earlier bound of $O(\sqrt{n})$ by Goaoc *et al.* [8].

1 Introduction

A *straight-line drawing* of a graph $G = (V, E)$ is a representation of G in the plane where the vertices V are mapped to distinct points in the plane, and each edge in E is mapped to a line segment between the corresponding vertices. A straight-line drawing is uniquely determined by an injective map $\pi : V \to \mathbb{R}^2$. A *geometric graph* is a graph $G = (V, E)$ together with a straight-line drawing $\pi : V \to \mathbb{R}^2$ in the plane. A straight-line drawing is *crossing-free* if no two edges intersect, except perhaps at a common endpoint. Every planar graph has a crossing-free straight-line drawing by Fáry's Theorem [7], however, not all straight-line drawings are crossing-free.

Suppose we are given a planar graph $G = (V, E)$ and a straight-line embedding $\pi : V \to \mathbb{R}^2$ (with possible edge crossings). The process of moving the vertices of G to new positions $\pi' : V \to \mathbb{R}^2$ to obtain a *crossing-free* straight-line drawing is called the *untangling* of (G, π). A vertex $v \in V$ is *fixed* in the untangling if $\pi(v) = \pi'(v)$.

In this paper we study the following problem: For an integer $n \in \mathbb{N}$, what is the maximum number $f(n)$ such that every planar geometric graph with n vertices can be untangled such that at least $f(n)$ vertices are fixed.

The first question on untangling planar geometric graphs was posed by Mamoru Watanabe in 1998: Is it true that every polygon P with n vertices can be untangled in at most ϵn steps, for some absolute constant $\epsilon < 1$, where in each step, we move a vertex of G to a new location. Watanabe's question was proved to be false by Pach and Tardos [14]. They showed that every polygon with n vertices can be untangled in at most $n - \sqrt{n}$ moves, and there are n-vertex polygons where no more than $O((n \log n)^{2/3})$ vertices can be fixed. Recently, Cibulka [5] proved that every n-vertex polygon can be untangled fixing $\Omega(n^{2/3})$ vertices,

The problem of untangling planar geometric graphs was studied by Goaoc *et al.* [8]. They proved $f(n) \leq \sqrt{n} + 2$ by constructing drawings of the planar graphs $P_2 * P_{n-2}$ with n vertices such that at most $\sqrt{n} + 2$ vertices are fixed in any untangling. Here P_k denotes a path with k vertices; and for two graphs, G and H, the join $G * H$ consists of the vertex-disjoint union of G and H and all

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edges between $V(G)$ and $V(H)$, see Fig. 1. Kang *et al.* [13] and Ravsky and Verbitsky [15] explored several families of n-vertex graphs where no more than $O(\sqrt{n})$ vertices can be fixed. Bose *et al.* [2] devised an algorithm untangles any geometric graph with n vertices while fixing $(n/3)^{1/4}$ vertices, which proves $f(n) \geq (n/3)^{1/4}$.

In this note, we improve the upper bound for $f(n)$ to $O(n^{1/(3-\log_{23} 22)}) \subset O(n^{.4965})$. We construct n-vertex planar geometric graphs for infinitely many values of $n \in \mathbb{N}$ such that any untangling fixes at most $O(n^{1/(3-\lambda)})$ vertices, where λ is the the shortness exponent of the family of 3-connected cubic planar graphs. The exact value of the shortness parameter λ is not known, the currently known best lower bound is $\lambda \geq \log_{23} 22 \approx 0.9858$ by Grünbaum and Walther [9]. Any improvement on the lower bound for λ would immediately improve our upper bound for $f(n)$.

Organization. In Section 2, we discuss two key ingredients of our construction: (i) the shortness exponent of cubic polyhedral graphs, and (ii) permutations with certain special properties related to the Erdős-Szekeres theorem. In Section 3, we present a family of planar geometric graphs and prove $f(n) \in O(n^{1/(3-\log_{23} 22)})$. We conclude in Section 4 by establishing a correspondence between the shortness parameter of cubic polyhedral graphs and the stabbing number of triangulations.

2 Preliminaries

Dual graphs of triangulations. The value of $f(n)$ is attained for edge-maximal planar graphs with n vertices, since a planar graph with more edges has fewer crossing-free straight-line embeddings. The edge-maximal planar graphs are called *triangulations*. Note that in every crossing-free drawing of an edge-maximal planar graph, every face (including the outer face) is bounded by three edges. It follows that every triangulation is 3-connected. By Euler's formula, a triangulation with $n \geq 3$ vertices has exactly $3n-6$ edges and and $2n-4$ faces (including the outer face). By Steinitz's theorem, every triangulation with $n \geq 4$ vertices is a polyhedral graph, that is, it is a 3-connected planar graph, and it is the 1-skeleton of a convex polytope in $\mathbb{R}^3$. Every polyhedral graph G has a well-defined dual graph G^* (independent of the plane embedding), corresponding to the 1-skeleton of the dual polytope. If G is a triangulation with $n \geq 4$ vertices, then G^* is a cubic polyhedral graph with $2n-4$ nodes and $3n-6$ edges.

Stabbing number of triangulations and dual cycles. The following observation is crucial for our construction.

Observation 1 *Let T be a polyhedral graph. Suppose that a line L stabs the faces $f_1, \ldots, f_k$ (in this order) in a crossing-free straight-line drawing of T. Then $(f_1^*, \ldots, f_k^*)$ is a simple cycle in the dual graph T^*.*

In Section 3, we will construct a planar graph G from two triangulations, S and T. Specifically, we plug a copy of S in each face of T. We then draw G in the plane such that the vertices of every copy of S are on a line L. If the dual graph T^* is not Hamiltonian, then in any crossing-free straight-line embedding of G, the line L will miss at least one face of T. If L misses a face f of T, then none of the vertices can be fixed in the copy of S plugged into f. In the next few paragraphs, we review the currently known best bounds on the maximum cycles in the dual graphs of triangulations.

In Section 4, we establish a somewhat surprising converse of Observation 1, and show that if $(f_1^*, \ldots, f_k^*)$ is a simple cycle in the dual graph T^* of a polyhedral graph T, then T has a crossing-free straight-line drawing such that a line L stabs the faces $f_1, \ldots, f_k$ in this order.

Maximum cycles in cubic polyhedral graphs. In an attempt at proving the Four Color Theorem, Tait [16] conjectured in 1884 that every cubic polyhedral graph is Hamiltonian. Tutte [18] found a counterexample with 44 vertices in 1946. The smallest known counterexample, due to Bernette, Bosák, and Lenderberg, has 38 vertices, and it is known that there is no counterexample with 36 or fewer vertices [10]. Using the smallest known counterexample to Tait's conjecture, one can build a cubic polyhedral graph with $\Theta(n)$ vertices for every $n \in \mathbb{N}$ in which every cycle has at most $O(n^{\log_{37} 36}) \subset O(n^{0.9925})$ vertices. Using similar techniques, Grünbaum and Walther [9] constructed a cubic polyhedral graph with $\Theta(n)$ vertices for every $n \in \mathbb{N}$ in which every cycle has at most $O(n^{\log_{23} 22}) \subset O(n^{0.9859})$ vertices.

The shortness exponents. The *shortness exponent* of a family of graphs was introduced by Grünbaum and Walther [9]. For a graph G, let $V(G)$ denote the set of vertices of G and let $h(G)$ be the number of vertices in a longest cycle in G (also known as the *circumference* of G). The shortness exponent of an infinite family $\mathcal{G}$ of graphs is

$$\lambda(\mathcal{G}) = \liminf \frac{\log h(G_n)}{\log |V(G_n)|}$$

over all infinite sequences of graphs $G_n \in \mathcal{G}$ where $\lim_{n\to\infty} |V(G_n)| = \infty$. This means that there are arbitrarily large graphs $G \in \mathcal{G}$ that contain a cycle of length $|V(G)|^{\lambda(\mathcal{G})-\varepsilon}$ for any fixed $\varepsilon > 0$.

For example, the shortness exponent is 1 for the family of Hamiltonian graphs, and 0 for the family of forests. The shortness exponent of cubic polyhedral graphs is not known. The currently best lower bound, due to Bilinski *et al.* [1], is $\lambda \geq x \approx 0.7532$, where x is the real root of $4^{1/x} - 3^{1/x} = 2$. The best upper bound is $\lambda \leq \log_{23} 22 \approx 0.9858$ due to Grünbaum and Walther [9].

Monotone subsequences. Erdős and Szekeres [6] showed that every permutation of $[n] = \{0, 1, \ldots, n-1\}$ contains a monotonically increasing or degreasing subsequence of length at least $\lceil\sqrt{n}\rceil$, and this bound is the best possible. The lower bound is attained on many different permutations. The best known construction consists of $\lceil\sqrt{n}\rceil$ monotonically increasing subsequences of consecutive elements, where the minimum element of each subsequence is larger than the maximum element of the next. We will use a permutation where the monotone sequences "spread out" more evenly. In a permutation $(\sigma_1, \sigma_2, \ldots, \sigma_n)$, we define the *spread* of a subsequence $(\sigma_{j_1}, \sigma_{j_2}, \ldots, \sigma_{j_k})$, $1 \leq j_1 < j_2 < \ldots. < j_k \leq n$, to be $j_k - j_1$.

Lemma 1 *For every $m \in \mathbb{N}$, there is a permutation π_n of $[n] = [4^m]$ such that*

- *the length of every monotone subsequence is at most $2^m = \sqrt{n}$; and*
- *the spread of every monotone subsequence of length $k \geq 2$ is at least $\frac{k^2+2}{6}$.*

Proof. We construct the permutation π_n by induction on m. For $m = 1$, let $\pi_4 = (2, 3, 0, 1)$ and observe that it has the desired properties. Assume that $\pi_n = (\sigma_1, \ldots, \sigma_n)$ is a permutation of $[n]$

with the desired properties. We construct a permutation π_{4n} of $[4n]$ by replacing each σ_i with the 4-tuple

$$(4\sigma_i + 2, 4\sigma_i + 3, 4\sigma_i + 0, 4\sigma_i + 1).$$

Let L be a monotone subsequence of length k in π_{4n}. Note that L has at most two elements from each 4-tuple. The sequence of these 4-tuples corresponds to a monotone subsequence of π_n, which we denote by L'. The length of L' is at least $k/2$, with equality iff L contains exactly two elements from each of the 4-tuples involved. By induction, the length of L' is $k/2 \leq 2^m$. Hence, we have $k \leq 2^{m+1}$, as required. If the length of L' is exactly $k/2$, then its spread is at least $\frac{(k/2)^2+2}{6}$ in π_n, and so the spread of L is at least $4(\frac{(k/2)^2+2}{6}) - 1 = \frac{k^2+2}{6}$. If the length of L' is more than $k/2$, then its spread is at least $\frac{(k/2+1)^2+2}{6}$, and the spread of L is at least $4(\frac{(k/2+1)^2+2}{6}) - 1 \geq \frac{k^2+2}{6}$, as required. □

3 Upper Bound Constructions

Theorem 1 *We have $f(n) \in O(n^{1/(3-\lambda)})$, where λ is the shortness exponent of the family of cubic polyhedral graphs.*

Proof. For every $n \in \mathbb{N}$, we construct planar graph $G = (V, E)$ with $\Theta(n)$ vertices and a straight line drawing $\pi : V \to \mathbb{R}^2$ such that in any untangling of G, at most $O(n^{1/(3-\lambda)})$ vertices are fixed. Let $\kappa = 1/(3 - \lambda)$.

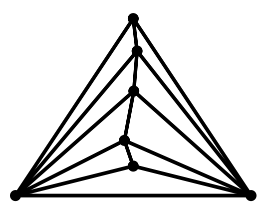

Figure 1: Triangulation $S = P_2 * P_5$.

Construction. We first construct the planar graph $G = (V, E)$. There is a cubic polyhedral graph T^* with $\Omega(n^\kappa)$ vertices such that every cycle in T^* has at most $O(n^{\kappa\lambda})$ vertices. The dual graph of T^* is a triangulation with $\Omega(n^\kappa)$ vertices such that in any crossing-free straight-line drawing of T, any line stabs at most $O(n^{\kappa\lambda})$ triangular faces.

Let S be the join $P_2 * P_{s+1}$ of two paths with 2 and $s+1$ vertices, respectively, where $s = \Theta(n^{1-\kappa})$ and s is a power of 4 (see Fig. 1). Note that S has exactly s interior vertices, which have a natural order along an interior path. We construct G by plugging in a copy of S into each face of T. Denote the copies of S by S_i, for $i = 1, 2, \ldots, \Theta(n^\kappa)$. The total number of vertices of G is $\Theta(n^\kappa + n^\kappa \cdot n^{1-\kappa}) = \Theta(n)$.

Next, we describe a straight-line drawing of G. Embed the vertices of the triangulation T arbitrarily in general position above the x-axis. Embed the interior vertices of S_1 into integer points $\{0, 1, \ldots, s-1\} \times \{0\}$ on the x-axis such that their natural order is permuted by π_s from Lemma 1. The interior vertices of S_i, for each $i > 1$, are embedded into a translated copy of this permutation, translated along the x-axis by δi for some small $0 < \delta \ll n^{-\kappa}$.

Bounding the number of fixed vertices. Consider a crossing-free straight-line drawing of G. The $\Theta(n^\kappa)$ vertices of T may be fixed. It is sufficient to consider the interior vertices of S_i, $i = 1, 2, \ldots, \Theta(n^\kappa)$. Suppose that ℓ_i interior vertices of S_i are fixed, for $i = 1, 2, \ldots, \Theta(n^\kappa)$. Since the x-axis intersects at most $O(n^{\kappa\lambda})$ triangles of T, all but at most $O(n^{\kappa\lambda})$ values of ℓ_i are zero.

Consider now a triangulation S_i where $\ell_i > 0$. Note that S_i contains a sequence of $s+1$ nested triangles that share a common edge (the horizontal edge in Fig. 1). In *any* straight-line drawing of S_i (independent of the choice of the outer face), at least $(s+1)/2$ of these triangles form a nested sequence. Hence, at least $\ell_i/2$ fixed interior vertices of S_i are vertices in a sequence of nested triangles in the crossing-free straight-line drawing of G. The intersection of the x-axis with a sequence of nested triangles is a line segment. It can be partitioned into two directed segments, with opposite directions, such that each of them is directed towards the deepest point in the arrangement of nested triangles. At least $\ell_i/4$ fixed points of S_i lie on the same directed segment, and these points must form a monotone sequence along the x-axis. Furthermore, the elements of this monotone subsequence are all contained in the largest triangle from the nested sequence of triangles in S_i, therefore, their convex hull is disjoint from the convex hulls of similar sequences in any other S_j, $j \neq i$.

By Lemma 1, the spread of the monotone subsequence of length at least $\ell_i/4$ is at least $(\ell_i^2 + 32)/96$. Hence these fixed points "occupy" an interval of length $(\ell_i^2 + 32)/96$ on the x-axis. As noted above, the convex hulls of monotone sequences from distinct copies of S are disjoint, and so we have

$$\sum_{i=1}^{\Theta(n^\kappa)} \frac{\ell_i^2 + 32}{96} \leq 2s. \tag{1}$$

Recall that at most $O(n^{\kappa\lambda})$ values of ℓ_i are nonzero. By Jensen's inequality, the sum $\sum_{i=1}^{\Theta(n^\kappa)} \ell_i$ is maximized if all nonzero values of ℓ_i are equal. Suppose, by relabeling the copies of S if necessary, that $\ell_i = \ell$ for $i = 1, 2, \ldots, \Theta(n^{\kappa\lambda})$; and $\ell_i = 0$ for all other i. In this case, Inequality (1) becomes $\Theta(n^{\kappa\lambda}) \cdot \ell^2 \leq \Theta(n^{1-\kappa})$, or $\ell \in O(n^{(1-\kappa(1+\lambda))/2})$. Therefore, the number of fixed vertices is at most

$$\sum_{i=1}^{\Theta(n^\kappa)} \ell_i \leq \Theta(n^{\kappa\lambda}) \cdot \ell = \Theta(n^{(1+\kappa(\lambda-1))/2}) = \Theta(n^\kappa),$$

as required. □

Combining Theorem 1 with the upper bound $\lambda \leq \log_{23} 22$ by Grünbaum and Walther [9], we obtain the following.

Corollary 1 $f(n) \in O(n^{1/(3-\log_{23} 22)}) \subset O(n^{.4965})$.

4 Stabbing number of triangulations

In this section, we prove the converse of Observation 1: if T is a polyhedral graph and $(f_1^*, \ldots, f_k^*)$ is a simple cycle in the dual graph T^*, then T has a crossing-free straight-line drawing such that a line stabs the faces $f_1, \ldots, f_k$ in this order. We construct the required straight-line embedding of T incrementally, based on the following two lemmas.

Recall that a *near-triangulation* is a planar graph such all faces are triangles with the possible exception of one face, which is considered to be the outer face. For example, every triangulation is a near-triangulation, where the outer face is also triangular. Tutte [17] proved that every near-triangulation has a straight-line embedding such that the outer face is mapped to a given convex polygon. This was extended by Hong and Nagamochi [11] to arbitrary star-shaped polygons (Lemma 2 below). Star-shaped polygons are defined in terms of visibility. Two points, p and q, are mutually *visible* with respect to a simple polygon P, if the a relative interior of the segment pq lies in the interior of P. The *kernel* of P, denoted $\ker(P)$, is the set of all points on the boundary and in the interior of f from which all vertices of f are visible. A polygon is *star-shaped* if it is has a non-empty kernel.

Lemma 2 *(Hong and Nagamochi [11]) Let G be a polyhedral graph where the outer face is bounded by a cycle with t vertices $(v_1, v_2, \ldots, v_t)$; and let $(p_1, p_2, \ldots, p_t)$ be a star-shaped polygon with k vertices. Then G has a crossing-free straight-line embedding $\pi : V \to \mathbb{R}^2$ such that $\pi(v_i) = p_i$ for $i = 1, 2, \ldots, t$.*

If T is a polyhedral graph embedded in the plane, then a simple cycle $C^*(f_1^*, \ldots, f_k^*)$ of the dual graph can be represented by a simple closed curve $\gamma = \gamma(C^*)$ that visits faces $f_1, \ldots, f_k$ of T in this order. For an inductive argument, it is convenient to work with such a closed curve γ in an arbitrary embedding of T.

Lemma 3 *Let $T = (V, E)$ be a 3-connected near-triangulation, and let $\pi : V \to \mathbb{R}^2$ be a crossing-free straight-line embedding of T such that the outer face is $(v_1, v_2, \ldots, v_t)$. Let γ be a closed Jordan curve that does not pass through any vertex of T and crossesk distince edges $(e_1, e_2, \ldots, e_k)$ in this order, where $e_1 = v_1v_2$ and $e_k = v_\tau v_{\tau+1}$ for some $2 \le \tau < t$. Let $P = (p_1, p_2, \ldots, p_k)$ be a star-shaped simple polygon such that a line L intersects the interior of $\ker(P)$ and crosses sides p_1p_2 and $p_\tau p_{\tau+1}$ (but no other side of P).*

Then T has a crossing-free straight-line drawing $\pi' : V \to \mathbb{R}^2$ such that $\pi'(v_i) = p_i$ for $i = 1, 2, \ldots, k$, and the edges crossed by line L are $e_1, \ldots e_k$ in this order.

Proof. We proceed by induction on k, the number of edges crossed by γ. Assume that $k \ge 3$, and Lemma 3 holds for any k, $3 \le k' < k$.

Refer to Fig. 2. Edges $e_1 = v_1v_2$ and e_2 are two sides of a triangle f_2, and so they have a common endpoint. Assume without loss of generality that $e_2 = v_2w$, with $w \ne v_1$. Denote by T_w the subgraph of T induced by the vertex set $\{v_1, v_2, \ldots, v_k, w\}$. The graph T_w consists of the chordless cycle $(v_1, v_2, \ldots, v_k)$, and a star between w and some vertices of $\{v_1, v_2, \ldots, v_k\}$ (including edges v_1w and v_2w). All bounded faces of T_w are incident to w, and they are each bounded by chordless cycles. Hence the subgraph of T lying in the interior or on the boundary of each bounded face of T_w is a 3-connected near-triangulation.

We are now ready to construct a crossing-free straight-line drawing π'. First, embed the vertices of T_w as follows. Let x be an intersection point of L and the interior of $\ker(P)$, and note that a small neighborhood of x is contained in $\ker(P)$. Let $\pi'(v_i) = p_i$ for $i = 1, \ldots, k$, and let $\pi'(w)$ be a point sufficiently close to x on the same side of line L as p_3. If w is sufficiently close to x, then all bounded faces of T_w are star-shaped, and whenever L crosses a bounded face of T_w, it also intersects the kernel of that face. Therefore, we can apply induction on the subgraphs of T lying in each bounded face F of T_w. If γ traverses a face of T that lies in the bounded face F of T_w, we can apply the induction hypothesis, otherwise we apply Lemma 2. □

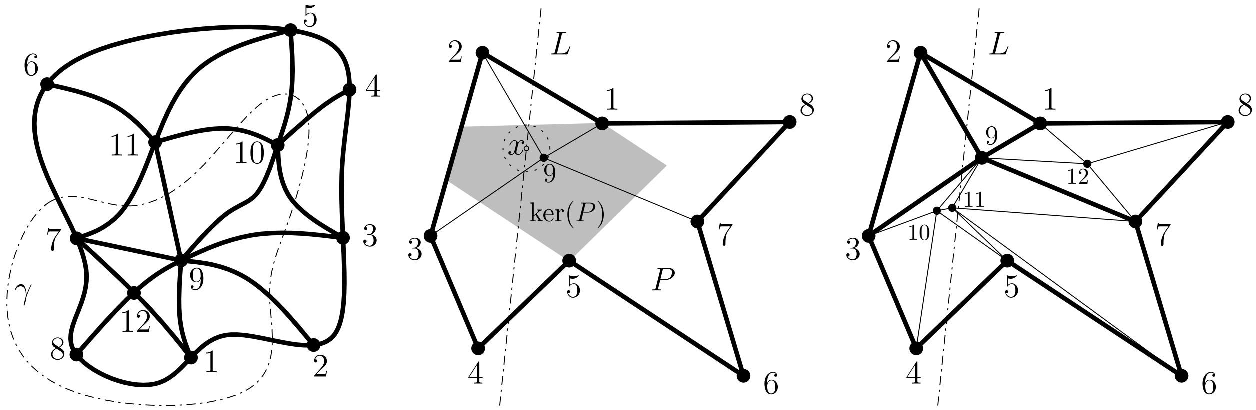

Figure 2: Left: a near-triangulation T, curve γ is a closed Jordan curve corresponding to simple cycle in the dual graph T^*. Middle: A star-shaped polygon P, with a shaded kernel $\ker(P)$. Vertex $w = v_9$ is embedded at a small neighborhood of a point $x \in L \cap \mathrm{int}(\ker(P))$. Right: we apply induction in each bounded face of T_w.

We are now ready to prove the converse of Observation 1.

Theorem 2 *Let $T = (V, E)$ be a polyhedral graph on $n \geq 4$ vertices and let $C^* = (f_1^*, \ldots, f_k^*)$ be a simple cycle in the dual graph T^*. Then T has a crossing-free straight-line drawing $\pi : V \to \mathbb{R}^2$ such that f_1 is the outer face.*

Proof. We are given a polyhedral graph $T = (V, E)$ and a simple cycle $C^* = (f_1^*, \ldots, f_k^*)$ in the dual graph T^*. Fix an arbitrary crossing-free straight-line drawing $\pi : V \to \mathbb{R}^2$ of T such that the outer face is f_1. Let γ be a closed Jordan curve that corresponds to the simple cycle $C^* = (f_1^*, \ldots, f_k^*)$, that is, γ traverses faces $f_1, \ldots, f_k$ in this order in the embedding π. Augment T with dummy edges to a near-triangulation T' by triangulating all bounded faces if necessary. We may assume that γ traverses every triangular face at most once. Denote the sequence of edges of T' crossed by γ by $e_1, \ldots, e_{k'}$, where e_1 and $e_{k'}$ are adjacent to the outer facer. If face f_1 has t vertices then let $P = (v_1, \ldots, v_t)$ be an arbitrary convex polygon with t vertices. By Lemma 3, T' has a crossing-free straight-line embedding such that the outer face is f_1 and a line L crosses the edges $e_1, \ldots, e_{k'}$ in this order. After deleting the dummy edges, we obtain a crossing-free straight-line embedding of T such that the outer face is f_1 and the line L stabs the faces $f_1, \ldots, f_k$ in this order, as required. □

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